NON-IMMERSION THEOREMS FOR REAL PROJECTIVE SPACES

By José Adem and Samuel Gitler

Introduction

Let ξ be the Hopf bundle over the n-dimensional real projective space RP^n. As is known ([12]), RP^n immerses in the euclidean space R^{n+k} if and only if the bundle $(n + k + 1)\xi$ has $n + 1$ independent non-zero sections.

The main result of this paper states that the bundle $(2n - 4)\xi$ does not have $n + 1$ independent non-zero sections if $n = 2^r + 2^s + 1$ with $r > s \geqq 2$ and $n > 16$. Therefore, for these values of n, we prove that RP^n does not have an immersion in R^{2n-5}. With the exception of $n = 15$, this includes our previous result in [4; Th. 12.3], since it trivially implies that RP^{n+2} does not have an immersion in $R^{2(n+2)-9}$. Now, Sanderson has shown in [13] that RP^{n+2} can be immersed in R^{2n-4}. Consequently, for n as before, we obtain that the three consecutive spaces RP^n, RP^{n+1} and RP^{n+2} have the best possible immersion in the same euclidean space R^{2n-4}.

The non-existence of $n + 1$ sections in the vector bundle $(2n - 4)\xi$ is established by computing some secondary cohomology operations in the Thom space of the bundle and using the fact that these operations vanish on low dimensional classes. The proof varies slightly according with the following three cases: $n = 2^r + 2^s + 1$ with $r > s + 1 \geqq 4$, $n = 2^r + 5$, and $n = 2^r + 2^{r-1} + 1$. For the first case the argument can be illustrated as follows. The Thom space of $(2n - 4)\xi$ is the stunted projective space RP^{3n-4}/RP^{2n-5}. The existence of $n + 1$ sections in $(2n - 4)\xi$ implies that RP^{3n-4}/RP^{2n-5} is the $(n + 1)$-fold suspension of a space whose mod 2 cohomology has trivial ring structure. With the use of stable secondary cohomology operations this is proved to be impossible.

The above results are extended to include the case $n = 2^r + 2$ with $n > 6$ which, with another method, was recently obtained by Baum and Browder in [6]. This paper also contains a simple proof of the non-immersion theorem of James ([10]) for $n = 2^r - 1$.

Unless otherwise stated, throughout this work we will use singular cohomology with coefficients Z_2, the cyclic group of order 2, and in general we will omit the coefficient group; thus $H^q(X)$ will stand for $H^q(X; Z_2)$.

Finally, we wish to thank E. H. Spanier and S. Y. Husseini for several suggestions that improved the presentation of this paper.

1. The twisted normal bundle

Let RP^n be the real projective n-space and ξ, the canonical line bundle over RP^n. An immersion of RP^n in R^{n+k}, the $(n + k)$-dimensional euclidean space, determines two k-plane bundles, the normal bundle ν and the *twisted normal bundle* $\nu \otimes \xi$. As is well known, the stable class of ν is independent of k, but this is not the case for $\nu \otimes \xi$. The twisted normal bundle has been used by Epstein and Schwarzenberger in [7] to construct embeddings of some real projective spaces and by Sanderson in [12] to give criteria for immersion.

Let $\varphi(n)$ be the number of integers r with $1 \leqq r \leqq n$ and $r \equiv 0, 1, 2, 4 \bmod 8$. For $n > 8$, the stable normal bundle of RP^n is $(2^{\varphi(n)} - (n+1))\xi$, and for an immersion of RP^n in R^{n+k}, the stable twisted normal bundle is $(n+k+1)\xi$.

THEOREM 1.1. *The following statements are equivalent*:

(1.2) RP^n *can be immersed in* R^{n+k}.

(1.3) *The bundle* $(2^{\varphi(n)} - (n+1))\xi$ *has* $2^{\varphi(n)} - (n+k+1)$ *independent non-zero sections.*

(1.4) *The bundle* $(n+k+1)\xi$ *has* $n+1$ *independent non-zero sections.*

Proof. The equivalence of (1.2) and (1.3) is Theorem (4.2) of [13]. The equivalence of (1.2) and (1.4) is Theorem (2.3) of [12]. For completeness we give a proof of this last equivalence. Let τ be the tangent bundle to RP^n. By the theorem of Hirsch ([8; Th. 6.1]), RP^n immerses in R^{n+k} if and only if there exists a k-vector bundle ν such that $\nu + \tau = n + k$. This in turn is equivalent with $\nu + \tau + 1 = n + k + 1$ by [11; Lemma 3.5]. Since $\tau + 1 = (n+1)\xi$, we have $\nu + (n+1)\xi = n + k + 1$. Now, using the fact that $\xi \otimes \xi = 1$ it follows that

$$\nu \otimes \xi + (n+1) = (n+k+1)\xi \tag{1.5}$$

if and only if RP^n immerses in R^{n+k}.

We now establish a corollary to (1.1) that will be used to obtain our non-immersion results. If α is a vector bundle over RP^n, let $(RP^n)^\alpha$ denote the Thom space of α. The following is shown by Atiyah in [5; (2.4), (4.3)]: If t is the trivial t-bundle then $(RP^n)^{\alpha+t} \cong S^t(RP^n)^\alpha$ where S^t is the t-fold suspension. Also, $(RP^n)^{r\xi} \cong RP^{n+r}/RP^{r-1}$ where the latter space denotes the stunted projective space.

COROLLARY 1.6. *If* ν *is the normal bundle of an immersion of* RP^n *in* R^{n+k}, *then*

$$S^t(RP^n)^\nu \cong RP^{2N-1}/RP^{2N-n-2}, \tag{1.7}$$

where $t = 2^{\varphi(n)} - n - k - 1$, $N = \varphi(n)$, *and*

$$S^{n+1}(RP^n)^{\nu\otimes\xi} \cong RP^{2n+k+1}/RP^{n+k}. \tag{1.8}$$

An immediate consequence of (1.5) is the following:

COROLLARY 1.9. *If* $x \in H^1(RP^n)$ *is the generator, then the total Stiefel-Whitney class of* $\nu \otimes \xi$ *is*:

$$W(\nu \otimes \xi) = (1 + x)^{n+k+1}.$$

If $2^r \leqq n < 2^{r+1}$ then the Stiefel-Whitney classes imply that RP^n cannot be immersed in $R^{2^{r+1}-2}$. If ν is the normal bundle of an immersion of RP^n in R^{n+k} with $k < n$, then $n + k > 2^{r+1} - 2$, and we have

PROPOSITION 1.10. $W_k(\nu \otimes \xi) \neq 0$ *if and only if* $n + 1 = 2^{r+1}$.

Proof. From (1.9) we have $W_k(\nu \otimes \xi) = \binom{n+k+1}{k} x^k$; and with the above hypothesis it follows easily that $\binom{n+k+1}{k} \equiv 1 \bmod 2$ if and only if $n + 1 = 2^{r+1}$.

2. The Gysin sequence

Let $\alpha = (E, X, \pi)$ be a k-vector bundle over X, with E the total space and $\pi : E \to X$ the projection. Let E_0 denote the subspace of E consisting of the non-zero vectors. We have the following commutative diagram,

$$\begin{array}{ccccccc} \to H^{q-1}(E) & \xrightarrow{i^*} & H^{q-1}(E_0) & \xrightarrow{\delta} & H^q(E, E_0) & \xrightarrow{j^*} & H^q(E) \\ & & & & {\scriptstyle T}\uparrow \wr & & \wr \uparrow {\scriptstyle \pi^*} \\ & & & & H^{q-k}(X) & \xrightarrow{\smile W_k} & H^q(X) \end{array} \tag{2.1}$$

where the upper sequence is the cohomology sequence of the pair (E, E_0), T is the Thom isomorphism, π^* is the isomorphism induced by the projection π, and W_k is the top Stiefel-Whitney class of α.

PROPOSITION 2.2. *Let X^α be the Thom space of α. If $W_k = 0$, the modulo two cohomology ring $H^*(X^\alpha)$ is a trivial ring.*

Proof. We have a natural isomorphism $H^*(X^\alpha) \approx H^*(E, E_0)$ in positive dimensions, so it is sufficient to prove that the ring $H^*(E, E_0)$ is trivial. Let $U \in H^k(E, E_0)$ be the Thom class of α. $W_k = 0$ implies that $U \smile U = \mathrm{Sq}^k U = 0$. Now if u and v are two elements of $H^*(E, E_0)$, by the Thom isomorphism there exist elements x and y in $H^*(X)$ such that $u = U \smile \pi^* x$ and $v = U \smile \pi^* y$. Consequently, $u \smile v = U \smile U \smile \pi^*(x \smile y) = 0$, and this ends the proof.

If $W_k = 0$, then $j^* = 0$ in (2.1) so that for all $q > 0$ we have the short exact sequences

$$0 \to H^{q-1}(E) \xrightarrow{i^*} H^{q-1}(E_0) \xrightarrow{\delta} H^q(E, E_0) \to 0. \tag{2.3}$$

Suppose now that $X = RP^n$. In this case, if $W_k = 0$ and $W_1 = 0$, then the Euler class $\chi(\alpha) = 0$ for n even or for n odd, and $k \neq n$. Therefore, with these hypotheses, (2.3) also holds over the integers.

Again, with $X = RP^n$, suppose that $W_k \neq 0$. Then $\smile W_k : H^{q-k}(RP^n) \to H^q(RP^n)$ is an isomorphism for all $k \leqq q \leqq n$. Therefore, from (2.1), it follows that in this range $j^* : H^q(E, E_0) \approx H^q(E)$. Consequently, if $h : RP^n \to (RP^n)^\alpha$ is the inclusion of the base in the Thom space, induced by the zero-section, we have

$$h^* : H^q((RP^n)^\alpha) \approx H^q(RP^n), \quad \text{for} \quad k \leqq q \leqq n. \tag{2.4}$$

3. The theorem of James

James in [10] has obtained strong non-immersion results for RP^n when $n + 1$ is a power of two. He uses axial maps and the reducibility of stunted projective spaces. We will obtain these results without using axial maps.

Suppose that RP^n immerses in R^{n+k} with normal bundle ν and twisted normal bundle $\nu \otimes \xi$. Since $(RP^n)^{\nu\otimes\xi}$ is $(k - 1)$-connected, the inclusion $h:RP^n \to (RP^n)^{\nu\otimes\xi}$ factors, up to homotopy, as shown in the following diagram,

$$\begin{tikzcd} RP^n \arrow[rr, "h"] \arrow[dr, "g"'] & & (RP^n)^{\nu\otimes\xi} \\ & RP^n/RP^{k-1} \arrow[ur, "f"'] & \end{tikzcd} \tag{3.1}$$

where g is the collapsing map. Then (2.4) and (3.1) give

$$f^*:H^q((RP^n)^{\nu\otimes\xi}) \approx H^q(RP^n/RP^{k-1}), \quad \text{for} \quad 0 \leqq q \leqq n. \tag{3.2}$$

Now, we recall from [10] that two spaces X and Y are mod 2 S-related if there exist integers r, s and a map $S^rX \to S^sY$ which induces an isomorphism in the homology mod 2.

PROPOSITION 3.3. *Let $n + 1$ be a power of two. If RP^n immerses in R^{n+k} then RP^n/RP^{k-1} and RP^{2n+1}/RP^{n+k} are mod 2 S-related.*

Proof. With the $(n + 1)$-fold suspension of the map f of (3.1) and the homeomorphism (1.8) we obtain a map

$$F:S^{n+1}(RP^n/RP^{k-1}) \to RP^{2n+k+1}/RP^{n+k}.$$

Since the first space in $(2n + 1)$-dimensional, this map factors, up to homotopy, as shown below,

$$\begin{tikzcd} S^{n+1}(RP^n/RP^{k-1}) \arrow[rr, "F"] \arrow[dr, "G"'] & & RP^{2n+k+1}/RP^{n+k} \\ & RP^{2n+1}/RP^{n+k} \arrow[ur, "i"'] & \end{tikzcd}$$

where i is the inclusion map. Consider the corresponding mod 2 cohomology diagram. From (3.2), F^* is an isomorphism for $0 \leqq q \leqq 2n + 1$. Also, i^* is an isomorphism in this range. Then G^* is an isomorphism for all q. Since the spaces are finite and the group of coefficients is the field Z_2, then G_*, the induced homomorphism in homology mod 2, is an isomorphism, and (3.3) follows.

THEOREM 3.4. (James). *If $n = 2^r - 1$, then RP^n cannot be immersed in R^{2n-q}, where*

$$\begin{aligned} q &= 2r && \textit{if} \quad r \equiv 1, 2 \ \textit{mod}\ 4 \\ q &= 2r + 1 && \textit{if} \quad r \equiv 0 \quad \textit{mod}\ 4 \\ q &= 2r + 2 && \textit{if} \quad r \equiv 3 \quad \textit{mod}\ 4. \end{aligned}$$

Proof. Suppose that RP^n can be immersed in R^{2n-q}. Then by (3.3), RP^n/RP^{n-q-1} and RP^{2n+1}/RP^{2n-q} are mod 2 S-related. Therefore, these two spaces are either both S-reducible or both not S-reducible by [10; (2.1) and footnote 4]. Now the results of Adams in [2] say that RP^{s+t}/RP^{s-1} is S-reducible if and only if $s + t + 1 \equiv 0 \bmod 2^{\varphi(t)}$. For the values of q in (3.4) one easily checks that RP^{2n+1}/RP^{2n-q} is S-reducible and that RP^n/RP^{n-q-1} is not S-reducible. This contradiction establishes the theorem.

4. Sphere bundles over real projective spaces

We will establish here some results about the action of the Steenrod squares Sq^i in the cohomology of sphere bundles over RP^n. These auxiliary results could be stated in a more general form, however, for simplicity, we restrict them to cover the needs of this paper.

THEOREM 4.1. *Let $\eta = (E, RP^n, \pi)$ be an orientable $4t$-vector bundle, and set $b = n - 4t + 1$. Suppose that $W_2 \neq 0$, $W_{4t} = 0$, $b \geqq 5$ and b not a power of two. If $U \in H^{4t}(E, E_0)$ is the Thom class of η, then there exists a unique class $u \in H^{4t-1}(E_0)$ such that $\delta u = U$ with $\mathrm{Sq}^{2i+1} u = 0$ and $\mathrm{Sq}^{2i} u = u \smile i^*\pi^* W_{2i}$ for all i.*

Proof. Since η is orientable, $W_1 = 0$ and, by the hypotheses, $W_{4t} = 0$ and $4t < n$. Therefore, from (2.3), the cohomology sequence of (E, E_0) breaks into short exact sequences over Z_2 and also over the integers Z. *If* $g^* = (\pi i)^*$, these sequences become

$$0 \to H^{q-1}(RP^n) \xrightarrow{g^*} H^{q-1}(E_0) \xrightarrow{\delta} H^q(E, E_0) \to 0.$$

If $q = 4t$, then $H^{4t-1}(RP^n, Z) = 0$ and there exists $u_1 \in H^{4t-1}(E_0\,; Z)$ such that $\delta u_1 = U_1$, where $U_1 \in H^{4t}(E, E_0\,; Z)$ is the integral Thom class of η. Then u and U are the reductions modulo 2 of u_1 and U_1, respectively, and we have $\delta u = U$ and $\mathrm{Sq}^1 u = 0$. These last two conditions guarantee the uniqueness of u.

From the Wu formulae we have $W_{2i+1} = \mathrm{Sq}^1 W_{2i} + W_1 W_{2i} = 0$. The action of the squares in the Thom class gives

$$\delta(\mathrm{Sq}^k u + u \smile g^* W_k) = 0. \tag{4.2}$$

This establishes the theorem for all $k > b$, since in this range the coboundary δ is an isomorphism.

We now consider the cases $k \leqq b$. From (4.2) it follows, in general, that

$$\mathrm{Sq}^k u = u \smile g^* W_k + a_k g^* x^{4t-1+k} \tag{4.3}$$

where $x \in H^1(RP^n)$ is the generator and a_k is either 0 or 1. To complete the proof, it is enough to show that $a_k = 0$ for all $0 \leqq k \leqq b$. To begin with, $a_0 = a_1 = 0$. Suppose $a_2 = 1$; then by applying Sq^2 to (4.3) with $k = 2$ we obtain $u \smile g^* W_2^2 + (\mathrm{Sq}^2 u) \smile g^* W_2 = 0$. Substitution of $\mathrm{Sq}^2 u$ for the expression given by the right side of (4.3) in this last equality gives $g^*(x^{4t+3}) = 0$; but this is a contradiction, since $n \geqq 4t + 4$. Therefore $a_2 = 0$. From $\mathrm{Sq}^3 u = \mathrm{Sq}^1 (u \smile g^* W_2) = 0$ it follows that $a_3 = 0$.

We proceed now by a four step induction. Suppose that $a_j = 0$ for $j = 0, 1, \cdots, 4i - 1$. Since the relation $\mathrm{Sq}^{4i+1} = \mathrm{Sq}^2\,\mathrm{Sq}^{4i-1} + \mathrm{Sq}^{4i}\,\mathrm{Sq}^1$ when applied to u gives $\mathrm{Sq}^{4i+1}\,u = 0$, it follows that $a_{4i+1} = 0$. If $4i < b$ and $a_{4i} = 1$, then by applying Sq^1 to (4.3), with $k = 4i$, we get $0 = \mathrm{Sq}^{4i+1}\,u = g^*(x^{4t+4i})$, which is a contradiction since $4t + 4i \leqq n$. Therefore, in this case $a_{4i} = 0$. Now, if $4i = b$, then $4i$ is not a power of two and Sq^{4i} is reducible, so there exists a relation of the form $\mathrm{Sq}^{4i} = \sum \mathrm{Sq}^r\,\mathrm{Sq}^s$ with $0 < r < 4i$. Using the induction hypothesis first, and then using it again together with the Cartan formula, we obtain

$$\mathrm{Sq}^{4i}\,u = \sum \mathrm{Sq}^r\,(u \smile g^*W_s) = u \smile g^*V,$$

where $V \in H^{4i}(RP^n)$. Applying δ we have $U \smile \pi^*W_{4i} = U \smile \pi^*V$, consequently, $W_{4i} = V$ and $a_{4i} = 0$.

By the preceding argument it also follows that $a_{4i+2} = 0$, since Sq^{4i+2} is reducible for $i > 0$. Finally, from $\mathrm{Sq}^{4i+3}\,u = \mathrm{Sq}^1(u \smile g^*W_{4i+2}) = 0$, we obtain $a_{4i+3} = 0$. This establishes the induction step and therefore the theorem.

5. Projective spaces under duality

We will use the definitions and notations of [14]. In particular, all the spaces under consideration are assumed to have a base point, the suspension is the reduced suspension, and $\{X, Y\}$ denotes the S-maps from X to Y. All the homology and cohomology groups are taken reduced, with coefficients in Z_2.

Let CP^a be the complex projective space of real dimension $2a$ and CP^a/CP^b the stunted projective space, where CP^b is identified with the base point.

THEOREM 5.1†. *Given RP^{2n+1}, there exists an integer $d > n$ and an S-map $\lambda \in \{CP^{n+d+1}/CP^d, S^{2d+1}RP^{2n+1}\}$ such that the induced homomorphism*

$$\lambda^*: H^{2q}(S^{2d+1}RP^{2n+1}) \to H^{2q}(CP^{n+d+1}/CP^d)$$

is an isomorphism for all q. Moreover, if $2^{k-1} \leqq n < 2^k$, then $d + 1 = g2^k$, for some integer g.

Proof. Let $X = RP^{2a}/RP^{2b+1}$ and $Y = CP^a/CP^b$. The canonical fibration $RP^{2a+1} \to CP^a$ induces a map $f: X \to Y$, which in cohomology gives isomorphisms in even dimensions. Let $\{f\} \in \{X, Y\}$ denote the S-map determined by f. With N large enough we take imbeddings $X, Y \subset S^N$ so as to obtain N-duals D_NX, D_NY. If $D_N: \{X, Y\} \to \{D_NY, D_NX\}$ is the Spanier-Whitehead homomorphism, we have the following commutative diagram:

$$\begin{array}{ccc}
H^q(Y) & \xrightarrow{f^*} & H^q(X) \\
\mathfrak{D}_N \downarrow & & \downarrow \mathfrak{D}_N \\
H_{N-1-q}(D_N Y) & \xrightarrow{\lambda_*} & H_{N-1-q}(D_N X)
\end{array}$$

† The existence of the map λ was pointed out to us by E. H. Spanier. For our purposes, we could have equally well used the map $\tilde{\omega}$ constructed by James in [9; p. 138].

where $\mathfrak{D}_N$ is the Alexander duality isomorphism and λ_* is the homomorphism induced by the S-map $\lambda = D_N\{f\}$. Clearly, λ_* is an isomorphism for q even.

Now, if $2a = 2^{\varphi(2n+1)} - 2$ and $2b = 2^{\varphi(2n+1)} - 2n - 4$, it follows from the Atiyah-James duality for projective spaces ([5; Th. 6.1]) that $D_N X$ is of the same S-type as RP^{2n+1}. Also, for some $d > n$, $D_N Y$ is of the same S-type as CP^{n+1+d}/CP^d. Since $N - 2b - 3$ is the highest dimension in which both $D_N X$ and $D_N Y$ have non-trivial cohomology, we have

$$\begin{aligned}\{D_N Y, D_N X\} &= \{S^{N-2b-3-2(n+1+d)}CP^{n+1+d}/CP^d, S^{N-2b-3-(2n+1)}RP^{2n+1}\}\\ &= \{CP^{n+1+d}/CP^d, S^{2d+1}RP^{2n+1}\},\end{aligned}$$

and this proves the existence of λ with the property that λ^* is an isomorphism in even dimensions.

Finally, to determine the form of d, *let* $x \in H^1(RP^{2n+1})$ and $\omega^{d+1} \in H^{2d+2}(CP^{n+1+d}/CP^d)$ be the generators. We have $\lambda^* S^{2d+1} x = \omega^{d+1}$, and, since the Steenrod squares commute with S-maps, we obtain $0 = \mathrm{Sq}^{2i}\lambda^* S^{2d+1}x = \mathrm{Sq}^{2i}\omega^{d+1} = \binom{d+1}{i}\omega^{d+1+i}$ for all $i > 0$. Therefore, $\binom{d+1}{i} \equiv 0 \bmod 2$ for all $i = 1, \cdots, n$, and consequently, $d + 1 = g2^k$ for some integer g.

6. Secondary cohomology operations

To prove non-immersion results for RP^n when $n + 1$ is not a power of two, we use secondary operations of two variables. They are a particular case of the stable secondary cohomology operations of several variables axiomatized by Adams; they are constructed from suitable pairs of relations in the Steenrod algebra A over Z_2.

Let

$$\alpha\beta = \textstyle\sum_{k=1}^m \alpha_k\beta_k = 0 \quad \text{and} \tag{6.1}$$

$$\alpha\theta = \textstyle\sum_{k=1}^m \alpha_k\theta_k = 0, \tag{6.2}$$

be two homogeneous relations in A of degrees $r + 1$ and r, respectively. The meaning of the composite operations $\alpha\beta = 0$ and $\alpha\theta = 0$ is as in [3; p. 98]. If t_k is the degree of θ_k, then $t_k + 1$ is the degree of β_k. In general, given a pair of relations, in order to write them in the form (6.1), (6.2) we allow some of the β_k and θ_j to be the zero operation. We suppose that $r > t_k \geqq 0$ where $t_k = 0$ is permissible only if θ_k is the zero operation.

With the above relations we construct a stable secondary cohomology operation Θ. This operation is defined on pairs of cohomology classes $u \in H^q(X)$ and $v \in H^{q+1}(X)$ which satisfy the condition $\beta(u) + \theta(v) = 0$ or, equivalently, $\beta_k(u) + \theta_k(v) = 0$ for $k = 1, \cdots, m$. It takes values on homogeneous cosets, precisely,

$$\Theta(u, v) \in H^{q+r}(X)/Q^{q+r}(\Theta, X),$$

where

$$Q^{q+r}(\Theta, X) = \textstyle\sum_{k=1}^m \alpha_k H^{q+t_k}(X). \tag{6.3}$$

The existence of the operation Θ is a consequence of the general result of Adams in [1; Th. 3.6.1], and, using his notation, all we need to do is exhibit a pair (d, z) constructed from the relations. To do this, let C_0 be the free graded left A-module generated by a_0 and a_1, where a_i is of degree i for $i = 0, 1$. Consider C_1 to be the free graded left A-module on m generators $b_1, \cdots, b_m$, with b_k of degree t_k. Define $d: C_1 \to C_0$ by $db_k = \beta_k(a_0) + \theta_k(a_1)$ on each generator. If we set $z = \sum_{k=1}^{m} \alpha_k b_k$, it follows from (6.1) and (6.2) that z is homogeneous of degree r, and $dz = 0$. Now, if u and v are cohomology classes as above, define $\epsilon: C_0 \to H^*(X)$ to be the left A-map of degree q given by $\epsilon(a_0) = u$, $\epsilon(a_1) = v$. Clearly the condition $\beta(u) + \theta(v) = 0$ is equivalent with $\epsilon d = 0$. Finally, the indeterminacy (6.3) follows from the expression for z.

The operation Θ satisfies the following Peterson-Stein type formula:

THEOREM 6.4. *Let $f: X \to Y$ be a map and $u \in H^q(Y)$, $v \in H^{q+1}(Y)$ be such that $f^*\beta(u) + f^*\theta(v) = 0$. Then the operations $\Theta(f^*u, f^*v)$ and $\alpha_f(\beta(u) + \theta(v))$ are defined, and $\Theta(f^*u, f^*v) = \alpha_f(\beta(u) + \theta(v)) \bmod Q^{q+r}(\Theta, X) + f^*H^{q+r}(Y)$.*

Here, the functional operation α_f is defined as in [3; p. 99]. The proof can be easily obtained along the same lines as in the proof of [4; Th. 10.8] and for this reason is omitted.

Now we will establish conditions for Θ to vanish because of dimensional reasons. If A is the Steenrod algebra over Z_2, let $A_k \subset A$ be the vector subspace of homogeneous elements of degree k and let $B(q)$ be the maximal left ideal of A which annihilates all cohomology classes of dimensions $\leqq q$ (see [15; p. 26]). Through relations (6.1) and (6.2), we define the integer $p = p(\Theta)$ to be the maximal number fulfilling the following conditions: either $\beta_k \in B(p)$ and $\theta_k \in B(p + 1)$, or $\alpha_k \in B(p + t_k + 1)$, for all $1 \leqq k \leqq m$. Let $H^*(X)$ be the cohomology algebra over the Steenrod algebra A. If $u \in H^q(X)$ and $v \in H^{q+1}(X)$, let $A[u, v] \subset H^*(X)$ be the subalgebra over A generated by u and v. Suppose $\beta(u) + \theta(v) = 0$, so that $\Theta(u, v)$ is defined.

THEOREM 6.5. *If $(A[u, v])^{q+r} = 0$, then $\Theta(u, v) = 0$ for all $q \leqq p(\Theta)$.*

Proof. Let $f: X \to K_1 \times K_2$ be a map such that $f^*\gamma_q = u$ and $f^*\gamma_{q+1} = v$, where $K_1 = K(Z_2, q)$ and $K_2 = K(Z_2, q + 1)$ are Eilenberg-MacLane spaces and γ_q, γ_{q+1} denotes the fundamental cohomology classes in $K_1 \times K_2$. Now, if $q \leqq p(\Theta)$, from (6.4) we obtain that (see [3; Th. 6.6])

$$\Theta(u, v) = \alpha_f(\beta(\gamma_q) + \theta(\gamma_{q+1})) = 0$$

with indeterminacy $Q^{q+r}(\Theta, X) + f^*H^{q+r}(K_1 \times K_2)$. But, $(A[u, v])^{q+r} = f^*H^{q+r}(K_1 \times K_2) = 0$; consequently $\Theta(u, v) = 0$ with its natural indeterminacy and this ends the proof.

7. The cohomology operations Φ_{2j} and Θ_{2j}

Let Φ_{2j}, with $j \geqq 2$, be the family of secondary cohomology operations considered in [4; §6]. We recall that these operations are associated with the single relations:

$$(7.1)\qquad \rho_{4k} = \mathrm{Sq}^1\,\mathrm{Sq}^{4k} + (\mathrm{Sq}^2\,\mathrm{Sq}^1)\,\mathrm{Sq}^{4k-2} + \mathrm{Sq}^{4k}\,\mathrm{Sq}^1 = 0, \quad \text{if } 2j = 4k,$$

$$(7.2)\qquad \rho_{4k+2} = (\mathrm{Sq}^2\,\mathrm{Sq}^1)\,\mathrm{Sq}^{4k} + \mathrm{Sq}^{4k+2}\,\mathrm{Sq}^1 = 0, \quad \text{if } 2j = 4k+2.$$

Let the operations Θ_{2j} of two variables be constructed using a pair of relations, according to the scheme of the preceding section. One of these relations will always be either (7.1) or (7.2), and the other relation will contain the term $\mathrm{Sq}^{2j}\,\mathrm{Sq}^2$. This second relation varies with the value of $2j$ modulo 8. We list below the four cases for the second relation:

$$(7.3)\qquad \sigma_{8k} = \mathrm{Sq}^2\,\mathrm{Sq}^{8k} + \mathrm{Sq}^4\,\mathrm{Sq}^{8k-2} + \mathrm{Sq}^{8k}\,\mathrm{Sq}^2 + \mathrm{Sq}^{8k+1}\,\mathrm{Sq}^1 = 0,$$

$$(7.4)\qquad \sigma_{8k+2} = \mathrm{Sq}^6\,\mathrm{Sq}^{8k-2} + \mathrm{Sq}^{8k+1}\,\mathrm{Sq}^3 + \mathrm{Sq}^{8k+2}\,\mathrm{Sq}^2 + \mathrm{Sq}^{8k+3}\,\mathrm{Sq}^1 = 0,$$

$$(7.5)\qquad \sigma_{8k+4} = (\mathrm{Sq}^4\,\mathrm{Sq}^2)\,\mathrm{Sq}^{8k} + \mathrm{Sq}^{8k+4}\,\mathrm{Sq}^2 + (\mathrm{Sq}^4\,\mathrm{Sq}^{8k+1})\,\mathrm{Sq}^1 = 0,$$

$$(7.6)\qquad \sigma_{8k+6} = \mathrm{Sq}^4\,\mathrm{Sq}^{8k+4} + \mathrm{Sq}^{8k+6}\,\mathrm{Sq}^2 + \mathrm{Sq}^{8k+7}\,\mathrm{Sq}^1 = 0.$$

Now, we define the operation Θ_{2j}, with $j \geqq 4$, using the pair of relations $\sigma_{2j} = 0$ and $\rho_{2j} = 0$ completed with the zero operations to take the form (6.1) and (6.2) respectively. For example, the operation Θ_{8k} is defined on pairs of cohomology classes $u \in H^q(X)$ and $v \in H^{q+1}(X)$ which satisfy $\mathrm{Sq}^2\,u + \mathrm{Sq}^1\,v = 0$ and $\mathrm{Sq}^i\,u = 0$, for $i = 1, 8k-2, 8k$, and $\mathrm{Sq}^j\,v = 0$, for $j = 8k-2$, 8k. It takes values in homogeneous cosets, namely

$$\Theta_{8k}(u, v) \in H^{q+8k+1}(X)/Q^{q+8k+1}(X)$$

where $Q^{q+8k+1}(X)$ is spanned by $\mathrm{Sq}^t\,H^{q+8k+1-t}(X)$ with $t = 1, 2, 4, 8k$.

The vanishing of the operations Φ_{2j} for dimensional reasons is given in [4; Th. 6.3]. The analogous result for Θ_{2j} is given by (6.5). A direct inspection of the pair of relations used gives

$$(7.7)\qquad p(\Theta_{2j}) = \begin{cases} 2j-4 & \text{if } \; j \equiv 0, 3 \bmod 4 \\ 2j-5 & \text{if } \; j \equiv 1, 2 \bmod 4. \end{cases}$$

A refinement of (6.5) which will be needed in the applications is the following:

THEOREM 7.8. *Let $u \in H^q(X)$ and $v \in H^{q+1}(X)$ be two classes such that $\Theta_{2j}(u, v)$ is defined, where $j \equiv 0, 3$ mod 4. If $(A[u, v])^{q+2j+1} = 0$ and $H^{q+2j-1}(X; Z_4) = 0$, then $\Theta_{2j}(u, v) = 0$ for all $q \leqq p(\Theta_{2j}) + 1$.*

Proof. We need consider only $q = p(\Theta_{2j}) + 1 = 2j - 3$. As in the proof of (6.5), let $f: X \to K$ be a map such that $f^*\gamma_q = u, f^*\gamma_{q+1} = v$, where $K = K_1 \times K_2$ and $K_1 = K(Z_2, q)$, $K_2 = K(Z_2, q+1)$. Since $(A[u, v])^{q+2j+1} = 0$, with the natural indeterminacy of Θ_{2j} we have, $\Theta_{2j}(u, v) = \alpha_f(\beta(\gamma_q) + \theta(\gamma_{q+1}))$. From the relations used for Θ_{2j} and the low dimensionality of u and v, the functional operation simplifies so that we obtain

$$\Theta_{2j}(u, v) = (\mathrm{Sq}^2\,\mathrm{Sq}^1)_f\,\mathrm{Sq}^{q+1}\,\gamma_{q+1} = \mathrm{Sq}^2\,(\mathrm{Sq}_f^1\,(\gamma_{q+1} \smile \gamma_{q+1})).$$

Now, in general, to compute $\mathrm{Sq}_f^1 w$ with $w \in H^{2q+2}(K)$ we proceed as follows (see [4; 7.8]). The condition $\mathrm{Sq}^1 w = 0$ implies the existence of $w_1 \in H^{2q+2}(K; Z_4)$ such that w is the modulo 2 reduction of w_1. The condition $f^* w = 0$ implies that $f^* w_1 = 2z_1$ for some $z_1 \in H^{2q+2}(X; Z_4)$, and the modulo 2 reduction of z_1 is a representative of $\mathrm{Sq}_f^1 w$. Let $w = \gamma_{q+1} \smile \gamma_{q+1}$. Since $H^{2q+2}(X; Z_4) = 0$ by hypothesis, we have $z_1 = 0$ and $\mathrm{Sq}_f^1 w = 0$. This ends the proof.

8. Cohomology operations in projective spaces

We begin this section by showing how the computation of Θ in the real projective space can be reduced to the computation of secondary operations of one variable in the complex projective space.

Given two homogeneous relations as (6.1) and (6.2), suppose that they are of degrees $2k + 2$ and $2k + 1$, respectively. Let Θ be an operation associated with this pair, and ϕ, a secondary operation of one variable associated with the single relation (6.2).

Let RP^∞ and CP^∞ be the infinite dimensional real and complex projective spaces and $x \in H^1(RP^\infty)$ and $\omega \in H^2(CP^\infty)$, the multiplicative generators. To determine the action of Θ in pairs (x^{2i}, x^{2i+1}), set $n = i + k$ and let d be an integer associated with RP^{2n+1} as in (5.1). Then we have:

PROPOSITION 8.1. *If $\Theta(x^{2i}, x^{2i+1})$ is defined, then $\Phi(\omega^{d+1+i})$ is defined and $\Theta(x^{2i}, x^{2i+1}) \neq 0$ if and only if $\Phi(\omega^{d+1+i}) \neq 0$.*

Proof. Restricting ourselves to RP^{2n+1}, let λ be the S-map of (5.1); then $\lambda^* S^{2d+1}(x^{2i}, x^{2i+1}) = (0, \omega^{d+1+i})$. The naturality and stability of Θ imply that $\Theta(0, \omega^{d+1+i})$ is defined, and with the same indeterminacy we have

$$\lambda^* S^{2d+1}\Theta(x^{2i}, x^{2i+1}) = \Theta(0, \omega^{d+1+i}) = \Phi(\omega^{d+1+i}).$$

Now, since $\lambda^* S^{2d+1}$ is an isomorphism in odd dimensions, the conclusion of (8.1) follows easily.

In view of the above result, to compute Θ_{2j} in pairs of cohomology classes of the real projective space, we will first compute the operations Φ_{2j} in some classes of the complex projective space.

Remark. The operation ϕ_{2j} is associated with the relation $\rho_{2j} = 0$ and has, in general, a smaller indeterminacy than the operation of one variable that one gets from the same relation $\rho_{2j} = 0$ completed with the zero operations so that $\sigma_{2j} = 0$ and $\rho_{2j} = 0$ give rise to the operation Θ_{2j}.

For m a positive integer, we denote by $\alpha(m)$ the number of non-zero terms in the dyadic expansion of m.

THEOREM 8.2. *Given Φ_{2j}, let $a = 2^r$ be such that $a \leqq 2j < 2a$. Let c be an integer which satisfies $2c < 2j - 2$ and $\alpha(c + j) > \alpha(c)$. Then for $t = ha + c$ with $h \geqq 1$, we have that $\Phi_{2j}(\omega^t)$ is defined, and with zero indeterminacy*

$$\Phi_{2j}(\omega^t) = h\binom{2c}{2j - a}\omega^{t+j}.$$

Proof. With $u = \omega^{ha}$, $v = \omega^{c}$, the hypotheses of the product formula in [4; Th. 6.4] can be verified. In fact, it is easily seen that $\Phi_{2j}(u)$ and $\Phi_{2j}(v)$ are defined; the conditions $A_k(u) = 0$ for $1 \leqq k \leqq 2j$ follow from $\alpha(2ha + k) > \alpha(2ha)$ (see [4; 8.2]); also $A_{2j}(v) = 0$ follows from the hypotheses $\alpha(c + j) > \alpha(c)$. The remaining conditions are trivially satisfied. Therefore, with zero indeterminacy, we have

$$\phi_{2j}(\omega^t) = \Phi_{2j}(u \smile v) = u \smile \Phi_{2j}(v) + \textstyle\sum_{k=0}^{j-2} \Phi_{2j-2k}(u) \smile \mathrm{Sq}^{2k}v.$$

Now, since $2c < 2j - 2$, from [4; Th. 6.3] we have $\Phi_{2j}(v) = 0$. Also, from [4; Th. 8.3] we have, for $0 \leqq k \leqq j - 2$, $\Phi_{2j-2k}(u) = 0$ if $2j - 2k \neq a$ and $\Phi_a(u) = h\omega^{ha+a/2}$. Consequently,

$$\Phi_{2j}(\omega^t) = \Phi_a(u) \smile \mathrm{Sq}^{2j-a}v = h\binom{2c}{2j - a}\omega^{t+j}.$$

THEOREM 8.3. *Given Θ_{2j}, let $a = 2^r$ be such that $a \leqq 2j < 2a$. Let c be an integer which satisfies $c \equiv 3 \bmod 4$, $2c < 2j - 6$ and $\alpha(c + j) > \alpha(c)$. Then for $t = ha + c$ with $h \geqq 1$, we have that $\Theta_{2j}(x^{2t}, x^{2t+1})$ is defined, and with zero indeterminacy*

$$\Theta_{2j}(x^{2t}, x^{2t+1}) = h\binom{2c}{2j - a}x^{2t+2j+1}.$$

Proof. It is straightforward to verify that under the given hypotheses, Θ_{2j} is defined and has zero indeterminacy. From (8.1), it follows that $\Phi_{2j}(\omega^{d+1+t})$ is defined and has zero indeterminacy where d is associated with $RP^{2t+2j+1}$ as in (5.1). Therefore $d + 1 = g2^k$ where $2^{k-1} \leqq t + j < 2^k$ and thus $d + 1 = 2ra$. All conditions of (8.2) are satisfied for $\Phi_{2j}(\omega^{d+1+t})$, and we have

$$\begin{aligned}\Phi_{2j}(\omega^{d+1+t}) &= \Phi_{2j}(\omega^{(2r+h)a+c}) = (2r + h)\binom{2c}{2j - a}\omega^{(2r+h)a+c+j} \\ &= h\binom{2c}{2j - a}\omega^{(2r+h)a+c+j}.\end{aligned}$$

Clearly, (8.3) follows from this equality and (8.1).

9. Non-immersion of real projective spaces

The non-immersion results will be obtained from the fact that some multiple of the Hopf bundle does not have enough independent non-zero sections.

THEOREM 9.1. *Let ξ be the Hopf bundle over RP^n. Then the bundle $(2n - 4)\xi$ does not have $n + 1$ independent non-zero sections in the following cases:*

(i) *if $n = 2^r + 2^s + 1$ with $r > s \geqq 2$ and $n > 16$;*

(ii) *if $n = 2^r + 2$ and $n > 6$.*

From (9.1) and Theorem 1.1, we immediately obtain the following:

THEOREM 9.2. *The real projective space RP^n cannot be immersed in R^{2n-5} if $n = 2^r + 2^s + 1$ with $r > s \geqq 2$ and $n > 16$, or if $n = 2^r + 2$ and $n > 6$.*

The result for $n = 2^r + 2$ is known and was established by Baum and Browder in [6], using a different technique. The case $n = 2^r + 2^s + 1$ implies our main non-immersion result of [4; Th. 12.3] for RP^m with $m = 2^r + 2^s + 3$, with the single exception of $m = 15$. However, the non-immersion result of RP^{15} is contained in the theorem of James (3.4).

Sanderson has proved in [13] that RP^{n+2} with $n = 4k + 1$ and k not a power of two can be immersed in R^{2n-4}. Combining this with (9.2) we have:

THEOREM 9.3. *The three consecutive real projective spaces* RP^n, RP^{n+1}, *and* RP^{n+2} *have the best possible immersion in the same euclidean space* R^{2n-4} *for all* $n > 16$ *of the form* $n = 2^r + 2^s + 1$ *with* $r > s \geqq 2$.

Proof of (9.1). We divide the proof of (i) into three cases.

Case 1: $r > s + 1 \geqq 4$. If $(2n - 4)\xi$ has $n + 1$ sections, $(2n - 4)\xi = n + 1 + \eta$ where $\eta = (E, RP^n, \pi)$ is an $(n - 5)$-vector bundle over RP^n. We have $W_{n-5}(\eta) = 0$ and from (2.2) it follows that $H^*((RP^n)^\eta)$ is a trivial ring.

The Thom space of $(2n - 4)\xi$ is $RP^{3n-4}/RP^{2n-5} \cong S^{n+1}(RP^n)^\eta$. If $U \in H^{n-5}(E, E_0)$ is the Thom class of η and x^{2n-4} and x^{2n-3} are the first two non-trivial classes of $H^*(RP^{3n-4}/RP^{2n-5})$, under the natural identification $H^*(E, E_0) \approx H^*((RP^n)^\eta)$, we have

$$x^{2n-4} = S^{n+1}U, \qquad x^{2n-3} = S^{n+1}(U \smile \pi^*x). \tag{9.4}$$

We will compute a secondary cohomology operation $\Theta_{2j}(x^{2n-4}, x^{2n-3})$ that leads to a contradiction. With the notation of (8.3), let $2j = 2^r + 2^s = n - 1$, $a = 2^r$, $c = 2^s - 1$, and $t = a + c = n - 2$. One easily verifies that the conditions of (8.3) are satisfied and that $\binom{2c}{2j - a} \equiv 1 \bmod 2$. Then $\Theta_{2j}(x^{2t}, x^{2t+1}) = x^{2t+2j+1}$. Therefore, by the stability of the operation, we have

$$\Theta_{2j}(U, U \smile \pi^*x) \neq 0.$$

On the other hand, from (7.7), we have that $n - 5 = p(\Theta_{2j})$ and $(A[U, U \smile \pi^*x])^{2n-5} = A_n(U) + A_{n-1}(U \smile \pi^*x)$, since $H^*((RP^n)^\eta)$ is a trivial ring. Now, $S^{n+1}A_n(U) = A_n(x^{2n-4})$ and $S^{n+1}A_{n-1}(U \smile \pi^*x) = A_{n-1}(x^{2n-3})$. From [4; 8.2] it follows that $A_n(x^{2n-4}) = 0$ and $A_{n-1}(x^{2n-3}) = 0$. Therefore, the hypotheses of (6.5) are satisfied, and we have $\Theta_{2j}(U, U \smile \pi^*x) = 0$. This contradiction establishes Case 1.

Case 2: $n = 2^r + 5$. If we suppose $(2n - 4)\xi$ has $n + 1$ sections, as in Case 1, we obtain an $(n - 5)$-vector bundle η with $W_{n-5}(\eta) = 0$, $W_1(\eta) = 0$, and $W_2(\eta) \neq 0$. Let x^{2n-4} and x^{2n-3} be as in (9.4). With $2j = 2^r + 4 = n - 1$, $a = 2^r$, $c = 3$, and $t = a + c = n - 2$, we verify the conditions of (8.3) and that $\binom{2c}{2j - a} \equiv 1 \bmod 2$. Then $\Theta_{2j}(x^{2t}, x^{2t+1}) = x^{2t+2j+1}$, and consequently $\Theta_{2j}(U, U \smile \pi^*x) \neq 0$.

In this case, in order to obtain a contradiction, we need to pass to $H^*(E_0)$ — the cohomology of the sphere bundle of η. One easily checks the hypotheses of (4.1) for η. Let $u \in H^{n-6}(E_0)$ be the unique class such that $\delta u = U$. If $v =$

$u \smile i^*\pi^*x$, we have $\delta v = U \smile \pi^*x$, $\Theta_{2j}(u, v)$ is defined and

$$\delta\Theta_{2j}(u, v) = \Theta_{2j}(U, U \smile \pi^*x).$$

Therefore, $\Theta_{2j}(u, v) \neq 0$.

Now, from (7.7), we have $n - 6 = p(\Theta_{2j})$, and from (4.1) it follows that $(A[u, v])^{2n-6} = A_n(u) + A_{n-1}(v)$. As in Case 1, we verify that $A_n(x^{2n-4}) = 0$ and $A_{n-1}(x^{2n-3}) = 0$, therefore $(A[u, v])^{2n-6} = 0$. Then, from (6.5) we have $\Theta_{2j}(u, v) = 0$. This contradiction establishes Case 2.

Case 3: $n = 2^r + 2^{r-1} + 1$. Let $p = 2^{\varphi(n)} - (n+1)$ and $q = 2^{\varphi(n)} - (2n - 4)$. From (1.1), to establish that $(2n-4)\xi$ does not have $n + 1$ sections it is enough to prove that $p\xi$ does not have q sections. If we suppose that $p\xi$ has q sections, then $p\xi = q + \eta$, where η is an $(n - 5)$-vector bundle over RP^n. We have $W_{n-5}(\eta) = 0$, $W_1(\eta) = 0$, and $W_2(\eta) \neq 0$. The Thom space of $p\xi$ is $RP^{n+p}/RP^{p-1} \cong S^q(RP^n)^\eta$. As before, we have $x^p = S^qU$ and $x^{p+1} = S^q(U \smile \pi^*x)$, where $U \in H^{n-5}(E, E_0)$ is the Thom class of η. Now, with $2j = 2^r + 2^{r-1} - 2 = n - 3$, $a = 2^r$, $c = 2^{r-2} - 1$, and $2t = 2ha + 2c = p$, where $h = 2^{\varphi(n)-r-1} - 1$, we verify the conditions of (8.3) and that $\binom{2c}{2j-a} = 1$. Then, $\Theta_{2j}(x^p, x^{p+1}) = x^{p+2j+1}$, and, consequently, $\Theta_{2j}(U, U \smile \pi^*x) \neq 0$. The vector bundle η satisfies the conditions of (4.1), and if $u \in H^{n-6}(E_0)$ is the class such that $\delta u = U$, with $v = u \smile i^*\pi^*x$, we have $\Theta_{2j}(u, v) \neq 0$.

Again, from (4.1), it follows that $(A[u, v])^{2n-8} = A_{n-2}(u) + A_{n-3}(v)$. Using [4; 8.2], we have $A_{n-2}(x^p) = 0$ and $A_{n-3}(x^{p+1}) = 0$; and then $(A[u, v])^{2n-8} = 0$. Since n is odd, we have $H^{n-4}(RP^n; Z_4) = 0$, and the Thom isomorphism implies $H^{2n-9}(E, E_0; Z_4) = 0$. From this and the cohomology sequence of the pair (E, E_0), we obtain $H^{2n-10}(E_0; Z_4) = 0$. Now, since $j \equiv 3 \bmod 4$ and $\dim u = n - 6 = p(\Theta_{2j}) + 1$, the conditions of (7.8) are satisfied and consequently $\Theta_{2j}(u, v) = 0$. This contradiction establishes the Case 3.

Finally, to prove (ii), if we suppose that $(2n - 4)\xi = n + 1 + \eta$ where η is an $(n - 5)$-vector bundle over RP^n, then $S^{n+1}(RP^n)^\eta \cong RP^{3n-4}/RP^{2n-5}$. We will show that this stunted projective space cannot be an $(n + 1)$-fold suspension when $n = 2^r + 2$ and $n > 6$.

Let Ψ_{2^r} be the secondary cohomology operation of one into two variables considered in [4; p. 73]. The first non-trivial cohomology class of RP^{3n-4}/RP^{2n-5} is x^{2n-4}, where $2n - 4 = 2^{r+1}$, and we have $x^{2n-4} = S^{n+1}U$ with U as in (9.4). From [4; Th. 11.5] we obtain that $\Psi_{2^r}(x^{2n-4}) \neq 0$. Therefore $\Psi_{2^r}(U) \neq 0$. But it is easy to verify that U satisfies the hypotheses of [4; Th. 11.8] and hence $\Psi_{2^r}(U) = 0$. This contradiction establishes (ii) and ends the proof of (9.1).

Centro de Investigación del I P N, México, D.F.

References

[1] J. F. Adams, *On the non-existence of elements of Hopf invariant one*, Ann. of Math., **72** (1960), 20–104.

[2] ———, *Vector fields on spheres*, Ann. of Math., **75** (1962), 603–32.

[3] J. Adem, *Sobre operaciones cohomológicas secundarias*, Bol. Soc. Mat. Mexicana, **7** (1962), 95–110.

[4] J. Adem and S. Gitler, *Secondary characteristic classes and the immersion problem*, Bol. Soc. Mat. Mexicana, **8** (1963) 53–78.

[5] M. F. Atiyah, *Thom complexes*, Proc. London Math. Soc., **11** (1961) 291–310.

[6] P. Baum and W. Browder, *The cohomology of quotients of classical groups*, Topology (to appear).

[7] D. B. A. Epstein and R. L. E. Schwarzenberger, *Imbeddings of real projective spaces*, Ann. of Math., **76** (1962), 180–84.

[8] M. W. Hirsch, *Immersions of manifolds*, Trans. Amer. Math. Soc., **93** (1959), 242–76.

[9] I. M. James, *Spaces associated with Stiefel manifolds*, Proc. London Math. Soc. (3), **9** (1959) 115–40.

[10] ———, *On the immersion problem for real projective spaces*, Bull. Amer. Math. Soc., **69** (1963), 331–38.

[11] M. A. Kervaire and J. W. Milnor, *Groups of homotopy spheres*, I, Ann. of Math., **77** (1963) 504–37.

[12] B. J. Sanderson, *A non-immersion theorem for real projective space*, Topology, **2** (1963) 209–11.

[13] ———, *Immersions and embeddings of projective spaces*, Proc. London Math. Soc. (3), **14** (1964) 135–53.

[14] E. H. Spanier, *Duality and the suspension category*, ("Symposium internacional de topología algebráica"), México, D.F., 1958.

[15] N. E. Steenrod and D. B. A. Epstein, *Cohomology operations*, Ann. of Math. Studies, No. 50, Princeton, 1962.