SOME IMMERSIONS ASSOCIATED WITH BILINEAR MAPS

BY JOSÉ ADEM

1. Introduction

As it is well known, the real projective space P^n has an immersion in the euclidean k-space R^k, if and only if there exists a nonsingular skew-linear map $f:R^{n+1} \times R^{n+1} \rightarrow R^{k+1}$, with $k > n$. Skew-linear means that $f(u, v)$ is linear in v and $f(-u, v) = -f(u, v)$, for all $u, v \in R^{n+1}$ (see [2], [3], [6]).

Does the existence of f imply the existence of a nonsingular bilinear map $g:R^{n+1} \times R^{n+1} \rightarrow R^{k+1}$? In general, this seems to be a very difficult question and it is not even known if it is possible to reduce the existence of g to a homotopy problem.

If such g exists, we say that the immersion of P^n in R^k is associated with a bilinear map. Clearly, it is enough to prove the existence of g for the best possible immersion, that is, for the minimal k.

K. Y. Lam proves in [5] that such g exists for all the immersions of P^n in R^k when $n \leq 15$.

In this paper, using some relations between associators and commutators in the Cayley algebra, we construct several nonsingular bilinear maps. Some of these maps allow us to extend the above result further. In fact, we get that all the immersions of P^n in R^k are associated with bilinear maps for $n \leq 23$, with the possible exception of $n = 19$.

2. Some properties of Cayley numbers

A nonassociative algebra K is *alternative* if $(xx)y = x(xy)$ and $(yx)x = y(xx)$ for all $x, y \in K$. Define the *commutator* and the *associator* by

$$[x, y] = xy - yx,$$

$$[x, y, z] = (xy)z - x(yz).$$

In an alternative algebra it follows that the associator is skew symmetric in its three variables ([9; p 27]). Then

$$[x, y, z] = -[x, z, y] = [y, z, x] = \cdots$$

Besides $[x, x, y] = 0$, $[x, y, x] = 0$ and $[y, x, x] = 0$, the associator of an alternative algebra satisfies many identities, as for example ([4; p 130])

$$[xy, z, x] = [x, y, z]x. \tag{2.1}$$

If we suppose that K is an alternative algebra without divisors of zero and of characteristic not 2, then we have

$$\text{if} \quad [x, y, z] \neq 0 \quad \text{then} \quad \begin{cases} [x, y] \neq 0, \\ [x, z] \neq 0, \\ [y, z] \neq 0. \end{cases} \tag{2.2}$$

This result is due to Bruck and Kleinfeld ([1; p 885]) and their proof will be given below for the case of the Cayley algebra. We will use this result as follows. If any of the commutators of (2.2) is zero, say

$$\text{(2.3)} \qquad \textit{if} \quad [x, y] = 0 \quad \textit{then} \quad [x, y, z] = 0.$$

From now on let K be the Cayley algebra over R, where R is the field of real numbers. We recall that K is an alternative algebra without zero divisors and that it has an involutorial antiautomorphism $x \to \bar{x}$, where $\bar{x}$ is the conjugate of x, satisfying

$$\overline{xy} = \bar{y}\bar{x}, \quad \bar{\bar{x}} = x \quad \text{for all} \quad x, y \in K.$$

Also, with the usual embedding $R \subset K$, we have that $t(x) \in R$ and $n(x) \in R$, where

$$t(x) = x + \bar{x} \quad \text{and} \quad n(x) = x\bar{x} = \bar{x}x,$$

are, respectively, the *trace* and the *norm* of x.

We can regard the elements of K as ordered pairs of quaternions. Given Cayley numbers $x = (a_1, a_2)$, $y = (b_1, b_2)$ where a_1, a_2, b_1, b_2, are quaternions, the product xy is given by

$$\text{(2.4)} \qquad xy = (a_1b_1 - \bar{b}_2a_2, b_2a_1 + a_2\bar{b}_1),$$

and the conjugate $\bar{x}$ of x by $\bar{x} = (\bar{a}_1, -a_2)$.

We will give now the proof of (2.2) directly for the Cayley algebra. Let $r \in R$ be defined by

$$r = n(x + y) - n(x) - n(y)$$

From the definition of trace and norm it is easy to verify that

$$xy + yx - yt(x) - xt(y) + r = 0.$$

Now, contrary to the conclusion of (2.2), suppose that $[x, y] = 0$, then $xy = yx$ and from the above equation, we obtain that $2xy - yt(x) = xt(y) - r$. Consequently,

$$[2xy - yt(x), z, x] = [xt(y) - r, z, x] = 0.$$

The equality with zero follows from the second associator, using the linear property together with the facts that K is alternative and that $r \in R$.

Then, from the first associator, equal to zero, using the linear property, the identity (2.1) and the skew symmetry, we obtain that

$$[x, y, z](2x - t(x)) = 0,$$

and since by hypothesis $[x, y, z] \neq 0$, we must have $2x - t(x) = 0$. But this implies that $x \in R$ and therefore the contradiction $[x, y, z] = 0$. This ends the proof.

In the Cayley algebra, besides the commutator $[x, y]$, we can define the *strong* *commutator* $\{x, y\}$ by

$$\{x, y\} = xy - \overline{xy}.$$

Observe that both commutators have real part zero (i.e. their traces are zero). For $\{x, y\}$ this is immediate and for $[x, y]$ follows from (2.4) as shown in [5].

We have that,

$$\{x, y\} = 0 \quad \text{implies} \quad [x, y] = 0.$$

The proof is as follows. We have that, $xy = \overline{xy}$ is a real number. Then, left multiplying this equation by $\bar{x}$, we get $n(x)y = \bar{x}\,\overline{xy} = \overline{xy}\,\bar{x}$, and right multiplying this result by x, we find $n(x)yx = \overline{xy}\,n(x)$. Consequently, $yx = \overline{xy} = xy$.

To emphasize the difference between the two types of commutators we will refer to $[x, y]$ as the *weak* commutator.

3. The map $K^3 \times K^3 \to K^5$

In order to obtain a map $K^3 \times K^3 \to K^5$ with a weak commutator in one of its components, the construction of $K^2 \times K^2 \to K^3$, given by K. Y. Lam in [5], will be extended one step further.

If $x_i, y_i \in K$, with $i = 1, 2, 3$, are Cayley numbers, let $u = (x_1, x_2, x_3)$ and $v = (y_1, y_2, y_3)$ denote the corresponding elements of K^3. Set

$$\Phi_1(u, v) = x_1y_1 + x_2y_2, \tag{3.1}$$

$$\Phi_2(u, v) = \bar{y}_1x_3 - x_1\bar{y}_3, \tag{3.2}$$

$$\Phi_3(u, v) = \bar{y}_1x_2 - x_1\bar{y}_2 + x_3y_3, \tag{3.3}$$

$$\Phi_4(u, v) = \bar{y}_2x_3 - x_2\bar{y}_3, \tag{3.4}$$

$$\Phi_5(u, v) = x_1y_1 - y_1x_1. \tag{3.5}$$

Then, define

$$f(u, v) = (\Phi_1(u, v), \cdots, \Phi_5(u, v)).$$

THEOREM 3.6. *The bilinear map $f: K^3 \times K^3 \to K^5$ is nonsingular. Moreover, by suitable restrictions, f induces the following nonsingular bilinear maps*:

(3.7) $R^{24} \times R^{24} \to R^{39}$, (3.8) $R^{21} \times R^{21} \to R^{35}$,

(3.9) $R^{19} \times R^{19} \to R^{33}$, (3.10) $R^{18} \times R^{18} \to R^{32}$,

(3.11) $R^{18} \times R^{24} \to R^{38}$, (3.12) $R^{19} \times R^{23} \to R^{37}$,

(3.13) $R^{18} \times R^{22} \to R^{36}$, (3.14) $R^{17} \times R^{24} \to R^{32}$.

Proof. If $f(u, v) = 0$, then $\Phi^k(u, v) = 0$ for $1 \leq k \leq 5$ and from the right-hand sides of (3.1–5), we obtain five equations that x_i, y_j must satisfy. To prove that f is nonsingular we use these equations and consider different cases.

First case: if $x_1 = 0$ and $y_1 = 0$. The system reduces to three equations and from these, using the fact that K has no zero divisors, it follows immediately that either $u = 0$ or $v = 0$.

Second case: if $x_1 = 0$ and $y_1 \neq 0$. Then, from (3.2), it follows that $x_3 = 0$, and with these values in (3.3) we obtain that $x_2 = 0$, therefore $u = 0$. The case $x_1 \neq 0$ and $y_1 = 0$, is settled in a similar way.

Third case: suppose that $x_1 \neq 0$ and $y_1 \neq 0$. It follows from (3.1) that we must have, in this case, that $x_2 \neq 0$ and $y_2 \neq 0$. Now, (3.2) implies that $x_3 = 0$ if and only if $y_3 = 0$. Then, we need to analyze two subcases: (1) when $x_3 = y_3 = 0$; (2) when all the x_i, y_j are different from zero.

If $x_3 = y_3 = 0$, then the system of equations reduces to

$$x_1y_1 + x_2y_2 = 0, \tag{3.15}$$

$$\bar{y}_1x_2 - x_1\bar{y}_2 = 0. \tag{3.16}$$

$$x_1y_1 - y_1x_1 = 0. \tag{3.17}$$

This is essentially the system used by Lam in his construction of the map $K^2 \times K^2 \to K^3$. Since we assume that $x_1 \neq 0$ and $y_1 \neq 0$, to find a contradiction, we proceed as follows. Left multiplying of (3.16) by y_1 and then using the associative property (2.3), gives

$$n(y_1)x_2 = y_1(x_1\bar{y}_2) = (y_1x_1)\bar{y}_2 = (x_1y_1)\bar{y}_2,$$

and now, right multiplying by y_2, we get

$$n(y_1)x_2y_2 = x_1y_1n(y_2). \tag{3.18}$$

From this and (3.15), we obtain

$$x_1y_1[n(y_1) + n(y_2)] = 0,$$

but this last result implies that $x_1y_1 = 0$, and this is a contradiction. Therefore, in this subcase we cannot have $f(u, v) = 0$ with $x_1 \neq 0$ and $y_1 \neq 0$.

To settle the second subcase we need some preliminary steps. Since $[x_1, y_1] = 0$, it follows that $[y_1, x_1y_1] = 0$, then

$$(y_1(x_1y_1))y_2 = ((x_1y_1)y_1)y_2 = (x_1y_1)(y_1y_2) \tag{3.19}$$

From (3.2) and (3.5), and with the same argument used to establish (3.18), we get

$$x_3y_3n(y_1) = x_1y_1n(y_3). \tag{3.20}$$

By substitution, from (3.3) and (3.20) we obtain

$$\bar{y}_1x_2 - x_1\bar{y}_2 + x_1y_1n(y_3)n(y_1)^{-1} = 0. \tag{3.21}$$

Now, left multiplying by y_1 and then right multiplying by y_2 in (3.21), and using (3.19), gives

$$n(y_1)x_2y_2 - x_1y_1n(y_2) + (x_1y_1)(y_1y_2)n(y_3)n(y_1)^{-1} = 0.$$

From (3.1) we have that $x_2y_2 = -x_1y_1$, then, by substitution, we obtain

$$\lambda x_1y_1 = (x_1y_1)(y_1y_2)n(y_3)n(y_1)^{-1},$$

where $\lambda = n(y_1) + n(y_2)$, and, by cancelling the factor x_1y_1, we get that

$$\lambda = y_1y_2n(y_3)n(y_1)^{-1}.$$

From this result it follows that

$$\bar{y}_1 = y_2n(y_3)\lambda^{-1} \quad \text{and} \quad \bar{y}_2 = y_1n(y_3)^{-1}\lambda, \tag{3.22}$$

and by taking norms, we also have that

$$\lambda^2n(y_3)^{-2} = n(y_2)n(y_1)^{-1}. \tag{3.23}$$

Substitution of (3.22) on (3.21), gives

$$y_2x_2 = x_1y_1(\lambda^2n(y_3)^{-2} - \lambda n(y_1)^{-1}),$$

and using (3.23) and the definition of λ proves that $y_2x_2 = -x_1y_1$. Therefore, from (3.1), we obtain that

$$y_2x_2 = x_2y_2 .$$

Now, the above commutativity can be used with (3.4) to establish

$$x_3y_3n(y_2) = x_2y_2n(y_3).$$

By adding this equation with (3.20) and using (3.1), it follows that $x_3y_3 = 0$. Therefore, this contradiction proves that we cannot have $f(u, v) = 0$, when all the numbers x_i, y_j are different from zero. This ends the proof that f is nonsingular.

To obtain the restrictions we proceed exactly as it is done in [5]. Let $x_1 = (a_1, a_2)$, $y = (b_1, b_2)$ be Cayley numbers represented as pairs of quaternions. From (2.4), it follows that

$$[x_1, y_1] = (a_1b_1 - b_1a_1 + \bar{a}_2b_2 - \bar{b}_2a_2, b_2(a_1 - \bar{a}_1) - a_2(b_1 - \bar{b}_1)). \tag{3.24}$$

As it was pointed out before, $[x_1, y_1]$ is purely imaginary, then it lies in a 7-dimensional subspace of K, and this gives (3.7). Let r_i, z_i and q_i denote, respectively, any real, any complex and any quaternion number. With (3.24) one easily verifies that the other maps are obtained by restricting x_1, y_1 as follows. For (3.8), $x_1 = (r_1, q_1)$, $y_1 = (r_2, q_2)$; for (3.9), $x_1 = (r_1, z_1)$, $y_1 = (r_2, z_2)$; for (3.10), $x_1 = (z_1, 0)$, $y_1 = (z_2, 0)$; for (3.11), $x_1 = (r_1, r_2)$, $y_1 = (q_1, q_2)$; for (3.12), $x_1 = (r_1, z_1)$, $y_1 = ((r_2, z_2), q_1)$; for (3.13), $x_1 = (z_1, 0)$, $y_1 = (z_2, q_1)$; for (3.14), $x_1 = (r_1, 0)$, $y_1 = (q_1, q_2)$.

Maps like (3.7) have been constructed in [7] and [8], but these maps do not give the restrictions (3.8–14).

With the notation given in the introduction, we have the following

COROLLARY 3.25 *All the immersions of P^n in R^k, where $n \leq 23$ and $n \neq 19$, are associated with bilinear maps.*

Proof. Up to $n \leq 15$ this follows from [5]. Now, with (3.10), P^{16} and P^{17} immerse in R^{31}; with (3.9), P^{18} immerses in R^{32}; with (3.8), P^{20} immerses in R^{34}; and with (3.7), P^{21}, P^{22} and P^{23} immerse in R^{38}. Since all these immersions are known to be best possible, this ends the proof.

4. The map $K^2 \times K^4 \to K^5$

In this section we construct a nonsingular bilinear map $R^{16} \times R^{32} \to R^{39}$, with a weak commutator in one of its components. This map has three interesting restrictions.

As before, let $u \in K^2$ and $v \in K^4$, where $u = (x_1, x_2)$ and $v = (y_1, y_2, y_3, y_4)$, with x_i, y_j Cayley numbers. Set

$$\Psi_1(u, v) = x_1 y_1 + x_2 \bar{y}_4, \tag{4.1}$$

$$\Psi_2(u, v) = x_2 y_2 - x_1 \bar{y}_3, \tag{4.2}$$

$$\Psi_3(u, v) = \bar{y}_1 x_2 - x_1 \bar{y}_2, \tag{4.3}$$

$$\Psi_4(u, v) = y_3 x_2 - x_1 y_4, \tag{4.4}$$

$$\Psi_5(u, v) = x_1 y_1 - y_1 x_1. \tag{4.5}$$

Now, define

$$g(u, v) = (\Psi_1(u, v), \cdots, \Psi_5(u, v)).$$

THEOREM 4.6. *The bilinear map $g: K^2 \times K^4 \to K^5$ is nonsingular. Furthermore, by suitable restrictions, g induces the following nonsingular bilinear maps*:

(4.7) $R^{16} \times R^{32} \to R^{39}$, (4.8) $R^{13} \times R^{29} \to R^{35}$,

(4.9) $R^{11} \times R^{27} \to R^{33}$, (4.10) $R^{11} \times R^{31} \to R^{37}$.

Proof. Suppose that $g(u, v) = 0$, then $\Psi_k(u, v) = 0$ for $1 \leq k \leq 5$, and from (4.1–5), we have five equations that x_i, y_j must satisfy. We consider three different cases.

First case: if $x_1 = 0$ and y_1 any number. In this case, it follows trivially that either $u = 0$ or $v = 0$.

Second case: if $x_1 \neq 0$ and $y_1 = 0$. Then, it follows, respectively, from (4.3), (4.2), (4.4) that $y_2 = 0$, $y_3 = 0$, $y_4 = 0$.

Third case: if $x_1 \neq 0$ and $y_1 \neq 0$. Then, (4.1) implies that $x_2 \neq 0$, $y_4 \neq 0$. From (4.4) it follows that $y_3 \neq 0$. Finally, (4.3) implies that $y_2 \neq 0$. So, in this case, all the x_i, y_j must be different from zero.

Now, like in (3.18), from (4.3) and (4.5), we get

$$x_2 y_2 n(y_1) = x_1 y_1 n(y_2).$$

From (4.1), we have that $x_1 y_1 = -x_2 \bar{y}_4$. If we substitute this in the above equation, and then cancel the factor x_2, we find that

$$y_2 n(y_1) = -\bar{y}_4 n(y_2). \tag{4.11}$$

Similarly, from (4.2), we have that $x_2y_2 = x_1\bar{y}_3$, and if we substitute this and cancel now the factor x_1, we obtain that

$$\bar{y}_3 n(y_1) = y_1 n(y_2). \tag{4.12}$$

If the expressions for y_3 and y_4, derived from (4.11) and (4.12) are replaced in (4.4), we get

$$\bar{y}_1 x_2 n(y_2)^2 = -x_1\bar{y}_2 n(y_1)^2.$$

From (4.3), we have that $\bar{y}_1 x_2 = x_1\bar{y}_2$, and combining these two results, gives

$$x_1\bar{y}_2(n(y_1)^2 + n(y_2)^2) = 0,$$

which is a contradiction to the fact that all the x_i, y_j are different from zero. Therefore, $g(u, v) = 0$ is not possible in this case, and this ends the proof that g is nonsingular.

The restrictions are obtained as in the previous theorem. Concretely, for (4.8), (4.9) and (4.10) proceed exactly as for (3.8), (3.9) and (3.12).

The other restrictions are not considered since they can be gotten from the Hurwitz-Radon maps.

Maps like (4.7) are constructed in [7] and [8]. The maps (4.8–10) are new.

5. The map $K^m \times K^m \to K^{2m-1}$

We will give a general construction of a nonsingular bilinear map $K^m \times K^m \to K^{2m-1}$, using strong commutators.

Let $(u, v) \in K^m \times K^m$, with $u = (x_1, \cdots, x_m)$ and $v = (y_1, \cdots, y_m)$. For $1 \leq p, q \leq m$, define

$$\Phi_{p,q}(u, v) = \begin{cases} \bar{y}_p x_q - x_p \bar{y}_q, & \text{if } p \neq q, \\ x_p y_p - \overline{x_p y_p}, & \text{if } p = q, \end{cases}$$

Now, set

$$\Psi_1(u, v) = \textstyle\sum_{p=1}^m x_p y_p, \tag{5.1}$$

$$\Psi_k(u, v) = \textstyle\sum \Phi_{p,q}(u, v), \tag{5.2}$$

where $2 \leq k \leq 2m - 1$, and the sum runs over all $p \leq q$ with $p + q = k$. Define

$$f(u, v) = (\Psi_1(u, v), \cdots, \Psi_{2m-1}(u, v)).$$

THEOREM 5.3. *The bilinear map* $f: K^m \times K^m \to K^{2m-1}$, *where* $m \geq 2$, *is nonsingular. Moreover, f induces the following nonsingular bilinear maps*:

$$R^{8n+2^k} \times R^{8n+2^k} \to R^{16n+2^k-1}, \tag{5.4}$$

where $n \geq 1$ *and* $0 \leq k \leq 3$.

Proof. By induction on m. True for $m = 2$, since in this case the system is as

(3.15–17), excepting that it has a strong commutator. Suppose that f is nonsingular for all $q < m$. As before, we consider different cases.

First case: if $x_1 = 0$ and $y_1 = 0$. Then, if we reindex the variables (i.e., $x_p' = x_{p+1}$ and $y_p' = y_{p+1}$), we get the map for $m - 1$ and, by the induction hypothesis, f is nonsingular.

Second case: if $x_1 = 0$, $y_1 \neq 0$ and $f(u, v) = 0$. We will prove by induction that $x_p = 0$ for all $1 \leq p \leq m$. Suppose that $x_1 = \cdots = x_t = 0$ for some $t < m$. With $k = t + 2$ we have $k \leq 2m - 1$, and the component $\Psi_k(u, v) = 0$ reduces to

$$\Psi_{t+2}(u, v) = \bar{y}_1 x_{t+1} = 0.$$

Since $y_1 \neq 0$, it follows that $x_{t+1} = 0$, and this completes the induction step. Therefore $u = 0$. The case $x_1 \neq 0$ and $y_1 = 0$ is settled in a similar form.

Third case: if $x_1 \neq 0$, $y_1 \neq 0$ and $f(u, v) = 0$. Here, we will arrive to a contradiction. Before, we will prove the following. For all $1 \leq p \leq m$ we have that, either (5.5) or (5.6) holds, where

$$x_p = y_p = 0, \tag{5.5}$$

$$x_1 y_1 n(y_p) = x_p y_p n(y_1) \neq 0. \tag{5.6}$$

This is again by induction. Suppose that it holds for all $1 \leq p \leq t$, with $t < m$. This induction hypothesis implies that

$$\Phi_{p,q}(u, v) = 0 \quad \text{for all} \quad p, q \leq t. \tag{5.7}$$

First, we will prove (5.7). From $\Psi_2(u, v) = x_1 y_1 - \overline{x_1 y_1} = 0$ and (5.6) it follows that $x_p y_p = \overline{x_p y_p}$. So, in either case (5.5) or (5.6), we have

$$\Phi_{p,p}(u, v) = x_p y_p - \overline{x_p y_p} = 0.$$

for all $p \leq t$, and this proves (5.7) for $p = q$. Also, from (2.5) it follows that

$$x_p y_p = y_p x_p \quad \text{for all} \quad p \leq t. \tag{5.8}$$

If $p \neq q$, with $p, q \leq t$, consider $\Phi_{p,q}(u, v) = \bar{y}_p x_q - x_p \bar{y}_q$. If $x_p = y_p = 0$ or $x_q = y_q = 0$, then $\Phi_{p,q}(u, v) = 0$. Otherwise, from (5.6) we have

$$x_1 y_1 n(y_p) = x_p y_p n(y_1),$$

$$x_1 y_1 n(y_q) = x_q y_q n(y_1),$$

and from these two equations, we get

$$x_p y_p n(y_q) = x_q y_q n(y_p).$$

With this and (5.8), reversing the steps of (3.18), we obtain

$$\Phi_{p,q}(u, v) = \bar{y}_p x_q - x_p \bar{y}_q = 0,$$

and this ends the proof of (5.7).

Going back to the induction step, consider

$$\Psi_{t+2}(u, v) = \sum \Phi_{p,q}(u, v) = 0,$$

where as in (5.2) the sum runs over all $p \leq q$ with $p + q = t + 2$. Substitute (5.7) to obtain

$$\Psi_{t+2}(u, v) = \bar{y}_{t+1}x_1 - x_{t+1}\bar{y}_1 = 0. \tag{5.9}$$

From $x_1 \neq 0$ and $y_1 \neq 0$, it follows that $x_{t+1} = 0$ if and only if $y_{t+1} = 0$. If $x_{t+1} \neq 0$ and $y_{t+1} \neq 0$, with $x_1y_1 = y_1x_1$ and (5.9), we obtain

$$(\bar{y}_{t+1}x_1)y_1 = \bar{y}_{t+1}(x_1y_1) = x_{t+1}n(y_1),$$

but $x_1y_1 = \overline{x_1y_1}$ is a real number, then

$$\bar{y}_{t+1}(x_1y_1) = (x_1y_1)\bar{y}_{t+1} = x_{t+1}n(y_1),$$

and from this, we get

$$x_1y_1n(y_{t+1}) = x_{t+1}y_{t+1}n(y_1),$$

so, the induction is completed, and either (5.5) or (5.6) holds for all $1 \leq p \leq m$.

To finish the proof, let

$$x_{r_i}y_{r_i} = x_1y_1n(y_{r_i})n(y_1)^{-1}, \qquad i = 1, \cdots, s \tag{5.10}$$

be all the terms such that (5.6) holds, and set

$$\lambda = \textstyle\sum_{i=1}^{s} n(y_{r_i})n(y_1)^{-1}$$

Then $\lambda \neq 0$ and by substitution on (5.1) of all the non zero terms (5.10), we get $\lambda x_1y_1 = 0$. Therefore, $x_1 = 0$ or $y_1 = 0$, contradicting the assumption. This ends the proof that f is nonsingular.

The induced maps are trivially obtained from the component $\Psi_2(u, v) = x_1y_1 - \overline{x_1y_1}$, by restricting x_1, y_1 to be a pair of real, complex and quaternion numbers.

Maps like (5.4) are constructed in [7] and [8].

With $n = 2^r$ and $k = 3$ the map (5.4) gives the best possible immersions of P^s, P^{s+1} and P^{s+2}, where $s = 2^{r+3} + 5$ and $r \geqq 0$.

CENTRO DE INVESTIGACIÓN DEL IPN

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