Boletín de la Sociedad Matemática Mexicana Vol. 37, 1992

CONSTRUCTIONS OF SUMS OF SQUARES FORMULAE WITH INTEGER COEFFICIENTS

BY TARA L. SMITH* AND PAUL YIU**

Dedicated to the memory of Professor José Adem

1. Introduction

This paper is on the construction of sums of squares formulae of the form

$$(x_1^2 + \cdots + x_r^2)(y_1^2 + \cdots + y_s^2) = z_1^2 + \cdots + z_n^2 \tag{1.1}$$

in which $z_1, \ldots, z_n$ are polynomials in $x_1, \ldots, x_r, y_1, \ldots, y_s$ with integer coefficients. We shall call an identity of this form an $[r, s, n]_{\mathbb{Z}}$ formula. $[r, s, n]_{\mathbb{Z}}$ formulae are generalizations of the classical $2-, 4-, 8-$square identities which express the multiplicative property of the norms of complex numbers, quaternions and octonions respectively. The impossibility of a $16-$square identity, *viz.* a $[16, 16, 16]_{\mathbb{Z}}$ formula, was discovered in the late 1840's. This suggested the problem of determining, for given r and s, the *smallest* integer n, denoted $r *_{\mathbb{Z}} s$, for which there exists an $[r, s, n]_{\mathbb{Z}}$ formula. The purpose of this paper is twofold: firstly as an announcement of the main theorem of [Y5] on the determination of the precise values of $r *_{\mathbb{Z}} s$ in the range $10 \leq r, s \leq 16$, and secondly to give some new upper bounds in the range $10 \leq r, s \leq 32$.

THEOREM (1). *The precise values of $r *_{\mathbb{Z}} s$ for $10 \leq r, s \leq 16$ are as follows.*

$r \setminus s$	10	11	12	13	14	15	16
10	16	26	26	27	27	28	28
11	26	26	26	28	28	30	30
12	26	26	26	28	30	32	32
13	27	28	28	28	32	32	32
14	27	28	30	32	32	32	32
15	28	30	32	32	32	32	32
16	28	30	32	32	32	32	32

The function $r *_{\mathbb{Z}} s$ is clearly symmetric in r and s. For $r \leq 9$, the values of $r *_{\mathbb{Z}} s$ are well known; see [S2, Section 4] or Theorem (9) below. That $10 *_{\mathbb{Z}} 10 = 16$ is an old result of K. Y. Lam [L2], and $16 *_{\mathbb{Z}} 16 = 32$ is the main result of [Y4]. A complete proof of Theorem (1) can be found in [Y5].

*Research partially supported by NSF
**Research partially supported by NSF Grant DMS 920-1204.

In Section 7 below, we shall exhibit sums of squares formulae realizing the above data. The constructions here are slightly different from those of [Y5]. Constructions of $[r, s, n]_{\mathbb{Z}}$ formulae beyond the range $r,\ s \leq 16$ have been considered in the works of Adem [A1], Yuzvinsky [Yuz 2], Lam and Smith [LS]. In Sections 8 and 9 below, we shall construct some new formulae in the range $10 \leq r, s \leq 32$, giving improved upper bounds of $r *_{\mathbb{Z}} s$.

THEOREM (2). *The table below gives upper bounds for $r *_{\mathbb{Z}} s$ in the range $r \leq s$, $10 \leq r \leq 32$, $17 \leq s \leq 32$. Those entries with asterisks give precise values of $r *_{\mathbb{Z}} s$.*

$r \setminus s$	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
10	29	29•	30	30	30*	30*•	32*	32*	32*	32*	32*	32*	32*	32*	32*	32*•
11	32	32	32	32	42	44	44	44	46•	48	48	48	48	52	52	52
12	32*	32*	32*	32*•	42•	44	44	44•	48	48	48	48	48•	52	52	52•
13	32*	32*	43	44	44	44•	48	48	48	48	48	59	60	60	60•	64
14	32*	32*	43•	44•	46•	48	48	48	48	48	48	59•	60•	62•	64	64
15	32*	32*	44	46	48	48	48	48	48	48	48	60	62	64	64	64
16	32*	32*	44•	46•	48	48	48	48	48	48	48•	60•	62•	64	64	64
17	32*	32*•	49•	50•	51•	52•	53•	54•	55•	56•	57•	61•	64	64	64	64
18		50	50•	52	52•	54	54•	56	56•	58•	61•	64	64	64	64	64
19			56	56	59•	60	60	64	64	64	64	64	64	64	64	64
20				56•	60	60	60•	64	64	64	64	64	64	64	64	64•
21					64	64	64	64•	75•	76•	80	80	80	84	84	84
22						72	72	72	76•	78	80	80	80•	84	84	84•
23							72	72	77•	78•	80	84•	88	92	92•	96
24								72•	78•	80	80	88	88•	94•	96	96
25									80	80	80	88•	94	96	96	96
26										80	80•	90•	94•	96	96	96
27											89•	93•	96	96	96	96
28												96	96	96	96	96
29													96	96	96	96
30														96	96	96•
31															116	116
32																116•

The proof of Theorem (2), to be found in Section 9, is by the exhibition of a formula of type indicated by each entry with a bullet in the table above. Other requisite formulae then follow from obvious restrictions.

D. B. Shapiro's paper [S2] and forthcoming book [S3] provide good surveys of results on $[r, s, n]$ formulae of the form (1.1) with $z_1, \ldots, z_n$ bilinear forms with coefficients from a given field. Recent results on the real coefficients case can be found in the works of K. Y. Lam and the second author [L3, L4, Y2, Y3, LY1, LY2, LY3], and for arbitrary fields in the work of Adem [A2, A3].

2. Preliminaries

It is well known that an $[r, s, n]_{\mathbb{Z}}$ formula is equivalent to a consistently signed intercalate matrix of type (r, s, n). We briefly recall the definitions. The matrices in the present paper are all combinatorial in nature. Let M be an $r \times s$ matrix with generic entry $M(i; j)$. We shall think of the entries of M as *colors*, and write $n(M)$ for the number of *distinct* colors in M.

Definitions (3). (a) An *intercalate matrix of type* (r, s, n), or simply an (r, s, n), is an $r \times s$ matrix M with $n(M) = n$ satisfying the following conditions.

(i) The colors along each row (respectively column) are distinct.
(ii) *Intercalacy*: If $M(i; j) = M(i'; j')$, then $M(i; j') = M(i'; j)$.

(b) An intercalate matrix M can be *signed consistently* if it is possible to endow each entry $M(i; j)$ with a sign $\epsilon_{i,j} = \pm 1$ such that

$$\epsilon_{i,j}\epsilon_{i,j'}\epsilon_{i',j}\epsilon_{i',j'} = -1 \quad \textit{whenever } M(i;j) = M(i';j') \quad \textit{for } i \neq i', j \neq j'. \tag{2.1}$$

Consider the *octonion algebra* $\mathbb{K}$ with identity e_1 and an orthonormal basis $e_1, e_2, \ldots, e_8$. The multiplication table with $(i; j)$ entry equal to $e_i e_j$ can be regarded as a consistently signed intercalate matrix with e_i replaced by "color" i, $1 \leq i \leq 8$:

$$\begin{bmatrix} +1 & +2 & +3 & +4 & +5 & +6 & +7 & +8 \\ +2 & -1 & +4 & -3 & +6 & -5 & -8 & +7 \\ +3 & -4 & -1 & +2 & +7 & +8 & -5 & -6 \\ +4 & +3 & -2 & -1 & +8 & -7 & +6 & -5 \\ +5 & -6 & -7 & -8 & -1 & +2 & +3 & +4 \\ +6 & +5 & -8 & +7 & -2 & -1 & -4 & +3 \\ +7 & +8 & +5 & -6 & -3 & +4 & -1 & -2 \\ +8 & -7 & +6 & +5 & -4 & -3 & +2 & -1 \end{bmatrix} \tag{2.2}$$

Let M_1 and M_2 be intercalate matrices of types (r, s_1, n_1) and (r, s_2, n_2) respectively, with *no* colors in common. Then the matrix $[\, M_1 \quad M_2 \,]$ is also intercalate, of type $(r, s_1 + s_2, n_1 + n_2)$. If M_1 and M_2 are each consistently signed, then so is M. Consequently, given $[r, s_1, n_1]_{\mathbb{Z}}$ and $[r, s_2, n_2]_{\mathbb{Z}}$ formulae, one obtains an $[r, s_1 + s_2, n_1 + n_2]_{\mathbb{Z}}$. We summarize this by writing

$$[r, s_1, n_1]_{\mathbb{Z}} \oplus [r, s_2, n_2]_{\mathbb{Z}} = [r, s_1 + s_2, n_1 + n_2]_{\mathbb{Z}}. \tag{2.3}$$

Likewise, two consistently signed intercalate matrices of types (r_1, s, n_1) and (r_2, s, n_2), containing *no* common colors, can be combined "vertically" to yield a consistently signed intercalate matrix of type $(r_1 + r_2, s, n_1 + n_2)$. This we shall summarize by writing

$$[r_1, s, n_1]_{\mathbb{Z}} \oplus' [r_2, s, n_2]_{\mathbb{Z}} = [r_1 + r_2, s, n_1 + n_2]_{\mathbb{Z}}. \tag{2.4}$$

For a positive integer k, we shall interpret $k[r, s, n]_{\mathbb{Z}}$ either as $[r, ks, kn]_{\mathbb{Z}}$ or $[kr, s, kn]_{\mathbb{Z}}$, as appropriate.

3. Dyadic intercalate matrices

It is easy to show ([Y3,Proposition 1.2]) that every intercalate matrix of type (n, n, n) must have $n = 2^t$ for some integer t, and is equivalent, up to permutation of rows and columns, and relabelling of colors, to the matrix D_t defined inductively by

$$D_{t+1} = \begin{pmatrix} D_t & 2^t + D_t \\ 2^t + D_t & D_t \end{pmatrix}, \qquad D_0 = (1). \tag{3.1}$$

Here, we think of D_t as a matrix of integers, and obtain $2^t + D_t$ by adding 2^t to *each* entry of D_t. Note that if $r, s \leq 2^t$, the $r \times s$ submatrix in the *upper left corner* of D_t is independent of t, and shall be denoted by $D_{r,s}$. Thus, for example, for $n = 2^t$, $D_{n,n} = D_t$. The intercalate matrices D_t play an important role in the construction of $[r, s, n]_{\mathbb{Z}}$ formulae.

One can think of D_t as the addition table of an elementary abelian 2−group. Let $\mathbb{N}$ be the set of positive integers. For each $n \in \mathbb{N}$, $n - 1$ is uniquely a finite sum of distinct powers of 2. Let $I(n)$ be the *finite* set of nonnegative integers satisfying

$$n - 1 = \sum_{i \in I(n)} 2^i. \tag{3.2}$$

We shall call $I(n)$ the *dyadic set* of n. Note that $I(1) = \emptyset$, the empty set. The relation

$$I(m \boxplus n) = (I(m) \setminus I(n)) \cup (I(n) \setminus I(m))$$

defines a binary operation $\boxplus$ on $\mathbb{N}$ that makes it into an abelian group with identity 1. This is an elementary abelian 2−group since $n \boxplus n = 1$ for each $n \in \mathbb{N}$. Note that

$$m \boxplus n = m + n - 1 \quad \text{if and only if} \quad I(m) \cap I(n) = \emptyset. \tag{3.3}$$

Conventional notations like $*$ and $\circ$ (see (3.4) below) being adopted in the present paper in the cardinal sense for enumeration purposes, we choose $\boxplus$ for the binary operation on $\mathbb{N}$ to reflect the ordinal aspect, the positions of colors in a standard intercalate matrix. The group structure on $\mathbb{N}$ is isomorphic to the infinite direct sum $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \cdots$, with addition of nonnegative integers in binary expansions without "carry over". We have shifted the integers by one unit to avoid using 0 as a color in an intercalate matrix. More importantly, this allows for elegant expressions for the numbers of colors in standard intercalate matrices. (See Proposition 7 below).

LEMMA (4). *(a)* $2^a(h \boxplus k - 1) + 1 = (2^a(h-1)+1) \boxplus (2^a(k-1)+1)$.
(b) If $k \leq 2^a$*, then for any* h, $\quad 2^a h + k = (2^a h + 1) \boxplus k$.

For $a, b \in \mathbb{N}$, denote $[a,b] = \{j \in \mathbb{N} : a \le j \le b\}$. For each integer $t \ge 0$, the subset $[1, 2^t]$ is a subgroup with addition table D_t:

$$D_t(m; n) = m \boxplus n \quad \text{for } 1 \le m, n \le 2^t.$$

More generally, $D_{r,s}$ can be regarded as the addition table of the subsets $[1, r]$ and $[1, s]$, and we shall call this the *dyadic intercalate matrix* of order $r \times s$. Denote by $r \circ s$ the number of *distinct* colors in the intercalate matrix $D_{r,s}$, *i.e.* $r \circ s = n(D_{r,s})$. It is easy to see that this can be determined recursively from

$$\text{(3.4)} \qquad \begin{aligned} &r \circ s = s \circ r, \quad 1 \circ s = s, \\ &r \circ s = \begin{cases} 2^{t-1} + r \circ (s - 2^{t-1}), & r \le 2^{t-1} < s, \\ 2^t, & 2^{t-1} < r, s \le 2^t. \end{cases} \end{aligned}$$

The number $r \circ s$ has also appeared in the work of A.Pfister on products of sums of squares in an arbitrary field of characteristic not 2. Let F be one such field. For each integer n, consider

$$D_F(n) = \{a \in F \setminus \{0\} : a \text{ is a sum of } n \text{ squares in } F\}.$$

Pfister [P] has established the following elegant theorem.

THEOREM (5). (Pfister) $D_F(r)D_F(s) = D_F(r \circ s)$.

There is an interesting relationship between the numbers $r \circ s$ and binomial coefficients. We recall a beautiful lemma discovered by E. Lucas in the nineteenth century, for the determination of the *parity* of binomial coefficients. See, for examples, [F] or [SE, Chapter I, Lemma 2.6]. Let n and m be given integers, say each smaller than 2^{t+1}. Write $n = \sum_{i=0}^{t} n_i 2^i$ for unique integers $n_i = 0$ or 1 in the range $0 \le i \le t$. Similarly, write $m = \sum_{i=0}^{t} m_i 2^i$. Then,

$$\binom{n}{m} \equiv \prod_{i=0}^{t} \binom{n_i}{m_i} \pmod 2.$$

Here, we interpret $\binom{0}{1} = 0$. In terms of dyadic sets, we have

$$\text{(3.5)} \qquad \binom{n}{m} \equiv 1 \pmod 2 \quad \text{if and only if} \quad I(m+1) \subseteq I(n+1).$$

LEMMA (6). *An integer $n + 1$ appears in $D_{r,s}$ if and only if there exists an integer j in the range $n - r < j < s$ satisfying $\binom{n}{j} \equiv 1 \pmod 2$.*

Proof. Suppose $n + 1 = h \boxplus k$ for $1 \le h \le r$, $1 \le k \le s$. Define integers h' and k' by

$$I(h') = I(h) \setminus I(k), \qquad I(k') = I(k) \setminus I(h).$$

Note that $h' \leq h$ and $k' \leq k$, and that $h' \boxplus k' = n+1$ since $I(h' \boxplus k') = I(h \boxplus k) = I(n+1)$. Furthermore, since $I(h') \cap I(k') = \emptyset$, we have, by (3.3), $n+1 = h' \boxplus k' = h' + k' - 1$ and

$$(n+1) - 1 = (h'-1) + (k'-1), \quad \text{with} \quad I(k') \subseteq I(n+1).$$

By the Lucas lemma, $\binom{n}{k'-1} \equiv 1 \pmod 2$, and $k'-1$ is in the range indicated in the statement of the proposition since

$$n - r \leq n - h < (n+1) - h' = k' - 1 < k' \leq k \leq s.$$

Conversely, suppose there is an integer j in the range $n - r < j < s$ such that the binomial coefficient $\binom{n}{j}$ is odd. This means that $I(j+1) \subseteq I(n+1)$, and $I(n-j+1) \cap I(j+1) = \emptyset$. From (3.3),

$$n+1 = (n-j+1) + (j+1) - 1 = (n-j+1) \boxplus (j+1) \in [1,r] \boxplus [1,s].$$

This completes the proof of the lemma. ∎

By a classic theorem of Hopf and Stiefel, the existence of an $[r,s,n]$ formula (with real coefficients) requires that

$$\binom{n}{j} \equiv 0 \pmod 2 \text{ for } n - r < j < s. \tag{3.6}$$

See, for example, [L3,p.175]. We shall, for convenience, refer to (3.6) as the *Hopf-Stiefel condition*. It is clear that if (r,s,n) satisfies the Hopf-Stiefel condition, then so does (r,s,m) for any $m \geq n$. The following corollary is immediate.

PROPOSITION (7). *(i)* $[1,r] \boxplus [1,s] = [1, r \circ s]$.
(ii) For given integers r and s, $r \circ s$ is the smallest integer n for which the triple (r,s,n) satisfies the Hopf-Stiefel condition.

Yuzvinsky [Yuz1] has conjectured that every intercalate matrix of order $r \times s$ contains *at least* $r \circ s$ colors. This conjecture, to the best of our knowledge, has hitherto been unresolved.

PROPOSITION (8). *If there is an $[r,s,r\circ s]_{\mathbb{Z}}$ formula, then $r *_{\mathbb{Z}} s = r \circ s$.*

Proof. The existence of an $[r,s,r\circ s]_{\mathbb{Z}}$ formula of course means that $r *_{\mathbb{Z}} s \leq r \circ s$. On the other hand, the triple $(r, s, r *_{\mathbb{Z}} s)$ must satisfy the Hopf-Stiefel condition, and $r *_{\mathbb{Z}} s \geq r \circ s$ by Proposition 7(ii). ∎

THEOREM (9). *For* $r \leq 9$ *or* $s \leq 9$, $r *_{\mathbb{Z}} s = r \circ s$.

Proof. First of all, note that the intercalate matrix $D_{9,16}$ can be signed consistently, for example, as in (6.3) below. Let $t \geq 4$ be an integer. The intercalate matrix $D_{9,2^t}$ can be regarded, in an obvious sense, as the *direct sum* of 2^{t-4} copies of $D_{9,16}$, each of which can be signed consistently. It follows that $D_{9,2^t}$ can be signed consistently, (even allowing $t \leq 3$). Given $r \leq 9$ and an arbitrary s, by choosing t satisfying $s \leq 2^t$, the intercalate matrix $D_{r,s}$, regarded as a submatrix of $D_{9,2^t}$, gives a formula of type $[r, s, r \circ s]_{\mathbb{Z}}$. By Proposition 8, $r *_{\mathbb{Z}} s = r \circ s$ for $r \leq 9$. The same equality holds for $s \leq 9$ by the symmetry of the functions $r *_{\mathbb{Z}} s$ and $r \circ s$. ∎

4. Formulae of the Hurwitz-Radon types

It is well known that given a positive integer $n = 2^t(2q+1)$, the *largest* positive integer r for the existence of an $[r, n, n]$ formula (with real coefficients) is given by the *Hurwitz-Radon number*

$$\rho(n) = \begin{cases} 2t+1, & t \equiv 0 \pmod 4, \\ 2t, & t \equiv 1, 2 \pmod 4, \\ 2t+2, & t \equiv 3 \pmod 4. \end{cases} \tag{4.1}$$

See, for example, [S1,S2]. If we write $t = 4a + b, 0 \leq b \leq 3$, then there is an alternative expression

$$\rho(n) = 8a + 2^b. \tag{4.2}$$

Since the classical works of Hurwitz [H] and Radon [R], many authors have written down different $[\rho(n), n, n]_{\mathbb{Z}}$ formulae in the form of matrices satisfying the Hurwitz-Radon equations. See, for examples, Wong [W], Zvengrowski [Z], Lam [L1], Geramita and Pullman [GP], Shapiro [S1], Yuzvinsky [Yuz1], Lam and Yiu [LY1]. All these involves only matrices with entries $0, \pm 1$. In other words, these all give formulae of type $[\rho(n), n, n]_{\mathbb{Z}}$. We shall call a $[\rho(n), n, n]_{\mathbb{Z}}$ formula one of the Hurwitz-Radon type. These formulae, however, exhibit different combinatorial structures. For example, a certain $[r, s, n]_{\mathbb{Z}}$ formula may arise as a restriction of one formula of the Hurwitz-Radon type, but not necessarily from another. We shall outline two constructions of Hurwitz-Radon type formulae by appropriately signing selected rows of the intercalate matrix D_t.

The first construction is due to Yuzvinsky [Yuz2], with correction by Lam and Smith [LS]. This can indeed be traced back to Eckmann [E]. Using the dyadic sets introduced in (3.2), we define the following functions.

$$L^o(k) = \text{Card}\{j \in I(k) : j \equiv 1 \pmod 2\},$$
$$L^e(k) = \text{Card}\{j \in I(k) : j \equiv 0 \pmod 2\}.$$

Also, for each integer i, let

$$L_i(k) = \text{Card}\{j \in I(k) : j \leq i\},$$
$$R_i(k) = \text{Card}\{j \in I(k) : j \geq i\}.$$

THEOREM (10). ([Yuz2, LS]) *Let t be a positive integer. Depending on the value of $t \pmod 4$, a $[\rho(2^t), 2^t, 2^t]_{\mathbb{Z}}$ formula can be constructed by signing the rows of the intercalate matrix D_t indicated in the following table, and endowing the entry in the h^{th} row and the k^{th} column with the sign $(-1)^{\lambda(h,k)}$, where λ is the function appearing in the rightmost column of the table.*

Row h	$t \equiv 0$	1	2	3 $\pmod 4$	$\lambda(h,k)$
1	✓	✓	✓	✓	0
$2^i + 1, 0 \leq i \leq t-1$	✓	✓	✓	✓	$L_i(k)$
$2^i + 2, 1 \leq i \leq t-2$	✓	✓	✓	✓	$R_i(k)$
$2^{t-1} + 2$		✓	✓	✓	$R_{t-1}(k) = \epsilon_{t-1}(k)$
$2^{t-1} - 1$	✓				$L^e(k) + \epsilon_{t-1}(k)$
2^{t-1}	✓				$1 + L^o(k) + \epsilon_{t-1}(k)$
$2^t - 1$				✓	$L^e(k)$
2^t				✓	$1 + L^o(k)$

For $t = 5$, for example, the above construction yields the following formula of type $[10, 32, 32]_{\mathbb{Z}}$:

(4.3)
$$\left[\begin{array}{ccccccccccccccc}
+1 & +2 & +3 & +4 & +5 & +6 & +7 & +8 & +9 & +10 & +11 & +12 & +13 & +14 & +15 \\
+2 & -1 & +4 & -3 & +6 & -5 & +8 & -7 & +10 & -9 & +12 & -11 & +14 & -13 & +16 \\
+3 & -4 & -1 & +2 & +7 & -8 & -5 & +6 & +11 & -12 & -9 & +10 & +15 & -16 & -13 \\
+4 & +3 & -2 & -1 & -8 & -7 & +6 & +5 & +12 & +11 & -10 & -9 & -16 & -15 & +14 \\
+5 & -6 & -7 & +8 & -1 & +2 & +3 & -4 & +13 & -14 & -15 & +16 & -9 & +10 & +11 \\
+6 & +5 & +8 & +7 & -2 & -1 & -4 & -3 & -14 & -13 & -16 & -15 & +10 & +9 & +12 \\
+9 & -10 & -11 & +12 & -13 & +14 & +15 & -16 & -1 & +2 & +3 & -4 & +5 & -6 & -7 \\
+10 & +9 & +12 & +11 & +14 & +13 & +16 & +15 & -2 & -1 & -4 & -3 & -6 & -5 & -8 \\
+17 & -18 & -19 & +20 & -21 & +22 & +23 & -24 & -25 & +26 & +27 & -28 & +29 & -30 & -31 \\
+18 & +17 & +20 & +19 & +22 & +21 & +24 & +23 & +26 & +25 & +28 & +27 & +30 & +29 & +32
\end{array}\right.$$

$$\left.\begin{array}{ccccccccccccccccc}
+16 & +17 & +18 & +19 & +20 & +21 & +22 & +23 & +24 & +25 & +26 & +27 & +28 & +29 & +30 & +31 & +32 \\
-15 & +18 & -17 & +20 & -19 & +22 & -21 & +24 & -23 & +26 & -25 & +28 & -27 & +30 & -29 & +32 & -31 \\
+14 & +19 & -20 & -17 & +18 & +23 & -24 & -21 & +22 & +27 & -28 & -25 & +26 & +31 & -32 & -29 & +30 \\
+13 & -20 & -19 & +18 & +17 & +24 & +23 & -22 & -21 & +28 & +27 & -26 & -25 & -32 & -31 & +30 & +29 \\
-12 & +21 & -22 & -23 & +24 & -17 & +18 & +19 & -20 & +29 & -30 & -31 & +32 & -25 & +26 & +27 & -28 \\
+11 & -22 & -21 & -24 & -23 & +18 & +17 & +20 & +19 & +30 & +29 & +32 & +31 & -26 & -25 & -28 & -27 \\
+8 & +25 & -26 & -27 & +28 & -29 & +30 & +31 & -32 & -17 & +18 & +19 & -20 & +21 & -22 & -23 & +24 \\
-7 & -26 & -25 & -28 & -27 & -30 & -29 & -32 & -31 & +18 & +17 & +20 & +19 & +22 & +21 & +24 & +23 \\
+32 & -1 & +2 & +3 & -4 & +5 & -6 & -7 & +8 & +9 & -10 & -11 & +12 & -13 & +14 & +15 & -16 \\
+31 & -2 & -1 & -4 & -3 & -6 & -5 & -8 & -7 & -10 & -9 & -12 & -11 & -14 & -13 & -16 & -15
\end{array}\right]$$

REMARK (11). Observe that by deleting the 10 columns in (4.3) containing the colors 31 and 32, we obtain a $(10, 22, 30)$ consistently signed. Further deleting the columns that contain also colors 29 and 30, we obtain a consistently signed $(10, 16, 28)$. Note that these explain the entries $(10; 16)$ in Theorem (1) and $(10; 22)$ in Theorem (2).

REMARK 12. Yuzvinsky [Yuz2], Lam and Smith [LS] have modified the above $[10, 32, 32]_{\mathbb{Z}}$ formula into a $[12, 20, 32]_{\mathbb{Z}}$ by appending two extra rows and deleting 12 columns to allow a consistent signing, as shown in (4.4) below. This explains the entry $(12; 20)$ in Theorem (2). Note that by deleting in (4.4) the 6 columns containing colors 23, 24, one obtains a $[12, 14, 30]_{\mathbb{Z}}$ formula, explaining the entries $(12; 14)$ and $(14; 12)$ of Theorem (1).

$$\begin{bmatrix}
+1 & +2 & +3 & +4 & +5 & +6 & +9 & +10 & +15 & +16 & +17 & +18 & +23 & +24 & +27 & +28 & +29 & +30 & +31 & +32 \\
+2 & -1 & +4 & -3 & +6 & -5 & +10 & -9 & +16 & -15 & +18 & -17 & +24 & -23 & +28 & -27 & +30 & -29 & +32 & -31 \\
+3 & -4 & -1 & +2 & +7 & -8 & +11 & -12 & -13 & +14 & +19 & -20 & -21 & +22 & -25 & +26 & +31 & -32 & -29 & +30 \\
+4 & +3 & -2 & -1 & -8 & -7 & +12 & +11 & +14 & +13 & -20 & -19 & -22 & -21 & -26 & -25 & -32 & -31 & +30 & +29 \\
+5 & -6 & -7 & +8 & -1 & +2 & +13 & -14 & +11 & -12 & +21 & -22 & +19 & -20 & -31 & +32 & -25 & +26 & +27 & -28 \\
+6 & +5 & +8 & +7 & -2 & -1 & -14 & -13 & +12 & +11 & -22 & -21 & +20 & +19 & +32 & +31 & -26 & -25 & -28 & -27 \\
+9 & -10 & -11 & +12 & -13 & +14 & -1 & +2 & -7 & +8 & +25 & -26 & +31 & -32 & +19 & -20 & +21 & -22 & -23 & +24 \\
+10 & +9 & +12 & +11 & +14 & +13 & -2 & -1 & -8 & -7 & -26 & -25 & -32 & -31 & +20 & +19 & +22 & +21 & +24 & +23 \\
+17 & -18 & -19 & +20 & -21 & +22 & -25 & +26 & -31 & +32 & -1 & +2 & -7 & +8 & -11 & +12 & -13 & +14 & +15 & -16 \\
+18 & +17 & +20 & +19 & +22 & +21 & +26 & +25 & +32 & +31 & -2 & -1 & -8 & -7 & -12 & -11 & -14 & -13 & -16 & -15 \\
+31 & -32 & +29 & -30 & -27 & +28 & +23 & -24 & +17 & -18 & -15 & +16 & -9 & +10 & +5 & -6 & -3 & +4 & -1 & +2 \\
+32 & +31 & -30 & -29 & +28 & +27 & -24 & -23 & -18 & -17 & +16 & +15 & +10 & +9 & -6 & -5 & +4 & +3 & -2 & -1
\end{bmatrix} \tag{4.4}$$

5. A doubling construction

Let an $[r, s, n]_{\mathbb{Z}}$ formula be represented by an intercalate matrix M of type (r, s, n), consistently signed, say with colors $c_1, c_2, \ldots, c_n$. Lam and Smith [LS] have discovered a simple construction leading to an $[r + 1, 2s, 2n]_{\mathbb{Z}}$ formula. Take an identical copy of M, and obtain another intercalate matrix M' of type (r, s, n) by (i) replacing each color c_i, $1 \le i \le n$ by a *new* color c_i', and (ii) changing the sign of each entry except those on the first row. Now to the $r \times 2s$ matrix $[\, M \quad M' \,]$, append a new row consisting of the s colors of the *first* row of M', *with the same signs,* followed by the s colors in the *first* row of M *with signs reversed.* In this way, one obtains a consistently signed intercalate matrix of type $(r+1, 2s, 2n)$. We shall call this the *doubling construction* and write

$$[r+1, 2s, 2n]_{\mathbb{Z}} = \mathfrak{D}[r, s, n]_{\mathbb{Z}}. \tag{5.1}$$

For example, the doubling construction on the $[8,8,8]_{\mathbb{Z}}$ formula given by (2.2) leads to a $[9,16,16]_{\mathbb{Z}}$, which we display in (6.3) below.

By interchanging the roles of r and s, we also obtain a formula of type $[2r, s+1, 2n]_{\mathbb{Z}}$:

$$[2r, s+1, 2n]_{\mathbb{Z}} = \mathfrak{D}'[r, s, n]_{\mathbb{Z}} \tag{5.2}$$

Adem [A1] has constructed formulae of types $[17,18,32]_{\mathbb{Z}}$ and $[18,17,32]_{\mathbb{Z}}$ by applying these constructions:

$$\begin{aligned}[17,18,32]_{\mathbb{Z}} &= \mathfrak{D}(\mathfrak{D}'[8,8,8]_{\mathbb{Z}}).\\ [18,17,32]_{\mathbb{Z}} &= \mathfrak{D}'(\mathfrak{D}[8,8,8]_{\mathbb{Z}}).\end{aligned} \tag{5.3}$$

We shall make frequent use of these constructions in the balance of this paper.

6. Another construction of formulae of the Hurwitz-Radon types

For $t = 4a+b$, $0 \le b \le 3$, let S_t be the set consisting of the integers

$$\begin{array}{ll} i, & 1 \le i \le 9, \\ 9 + 8\sum_{j=1}^{l-1} 16^j + i \cdot 16^l, & 1 \le i \le 8,\ 1 \le l \le a-1, \\ 9 + 8\sum_{j=1}^{a-1} 16^j + i \cdot 16^a, & 1 \le i \le 2^b - 1. \end{array}$$

Note that S_t contains precisely $\rho(2^t) = 8a + 2^b$ integers. For example,

$$S_5 = [1,9] \cup \{25\}. \tag{6.2}$$

THEOREM (13). *The submatrix of $D_t = D_{2^t,2^t}$ consisting of rows $h, h \in S_t$, can be signed consistently to give a formula of type $[\rho(2^t), 2^t, 2^t]_{\mathbb{Z}}$.*

We shall only indicate the signing of the matrix. A detailed analysis can be found in [Y1]. Such an analysis would also demonstrate that the doubling construction explained in Section 5 can be understood in the more general context of formulae with *real* coefficients.

For $t \leq 3$, the intercalate matrix in question is signed according to (2.2). For $t = 4$, $S_4 = [1, 9]$. We perform the doubling construction to the $[8, 8, 8]_{\mathbb{Z}}$ formula and obtain the following $[9, 16, 16]_{\mathbb{Z}}$:

(6.3)

$$\begin{bmatrix}
+1 & +2 & +3 & +4 & +5 & +6 & +7 & +8 & +9 & +10 & +11 & +12 & +13 & +14 & +15 & +16 \\
+2 & -1 & +4 & -3 & +6 & -5 & -8 & +7 & -10 & +9 & -12 & +11 & -14 & +13 & +16 & -15 \\
+3 & -4 & -1 & +2 & +7 & +8 & -5 & -6 & -11 & +12 & +9 & -10 & -15 & -16 & +13 & +14 \\
+4 & +3 & -2 & -1 & +8 & -7 & +6 & -5 & -12 & -11 & +10 & +9 & -16 & +15 & -14 & +13 \\
+5 & -6 & -7 & -8 & -1 & +2 & +3 & +4 & -13 & +14 & +15 & +16 & +9 & -10 & -11 & -12 \\
+6 & +5 & -8 & +7 & -2 & -1 & -4 & +3 & -14 & -13 & +16 & -15 & +10 & +9 & +12 & -11 \\
+7 & +8 & +5 & -6 & -3 & +4 & -1 & -2 & -15 & -16 & -13 & +14 & +11 & -12 & +9 & +10 \\
+8 & -7 & +6 & +5 & -4 & -3 & +2 & -1 & -16 & +15 & -14 & -13 & +12 & +11 & -10 & +9 \\
+9 & +10 & +11 & +12 & +13 & +14 & +15 & +16 & -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8
\end{bmatrix}$$

For $1 \leq h \leq 9$ and $1 \leq k \leq 16$, denote by $\epsilon_{h,k}$ the *sign* $= \pm 1$ in the $(h; k)$ entry of (6.3). For $t \geq 5$, we sign the rows of D_t specified by S_t in (6.1) as follows.

(1) The signs along the first row are all $+1$.
(2) For each $i = 2, \ldots, 9$, repeat the sequence of 16 signs in row i of (6.3) 2^{t-4} times to form a string of 2^t signs.
(3) For each $l = 1, \ldots, a-1$, $\quad i = 1, \ldots, 8$, construct a sequence of 16^{l+1} signs as follows: for each $1 \leq j \leq 16$, the j^{th} block consists of 16^l signs, each equal to $\epsilon_{i+1,j}$. Repeat this block $2^{t-4(l+1)}$ times to form a string of 2^t signs for row $9 + 8\sum_{j=1}^{l-1} 16^j + i \cdot 16^l$.
(4) For each $i = 1, \ldots, 2^b - 1$, and for the row $9 + 8\sum_{j=1}^{a-1} 16^j + i \cdot 16^a$, form a string of 2^b blocks of signs, each block of length 16^a, the signs in the j^{th} block, $j = 1, \ldots, 2^b$, being all $\epsilon_{i+1,j}$.

7. Construction of formulae realizing the data of Theorem (1)

We construct $[r, s, n]_{\mathbb{Z}}$ formulae realizing the data in Theorem (1). The more difficult task of justifying that these data give the precise values of $r *_{\mathbb{Z}} s$ in the range $10 \leq r, s \leq 16$ requires a detailed analysis of the structure of intercalate matrices, and can be found in [Y5]. We shall be content with a few remarks. This analysis is based on the recognition that intercalate matrices of certain partition patterns cannot be signed consistently. Essential use is also made of the notion of *hidden formulae* discovered in the general context of quadratic forms between euclidean spheres. [Y2,LY2]. It is also interesting to point out that these data are not analyzed individually, but are rather treated simultaneously.

7A. As remarked before, $10 *_{\mathbb{Z}} 10 = 16$ is well known. (7.1) is below is a consistent signing of $D_{10,10}$, giving $10 *_{\mathbb{Z}} 10 \leq 16$. Equality follows from Proposition 8 since $10 \circ 10 = 16$.

$$
\begin{bmatrix}
+1 & +2 & +3 & +4 & +5 & +6 & +7 & +8 & +9 & +10 \\
+2 & -1 & +4 & -3 & +6 & -5 & -8 & +7 & +10 & -9 \\
+3 & -4 & -1 & +2 & +7 & +8 & -5 & -6 & +11 & +12 \\
+4 & +3 & -2 & -1 & +8 & -7 & +6 & -5 & +12 & -11 \\
+5 & -6 & -7 & -8 & -1 & +2 & +3 & +4 & +13 & +14 \\
+6 & +5 & -8 & +7 & -2 & -1 & -4 & +3 & +14 & -13 \\
+7 & +8 & +5 & -6 & -3 & +4 & -1 & -2 & +15 & -16 \\
+8 & -7 & +6 & +5 & -4 & -3 & +2 & -1 & +16 & +15 \\
+9 & -10 & -11 & -12 & -13 & -14 & -15 & -16 & -1 & +2 \\
+10 & +9 & -12 & +11 & -14 & +13 & +16 & -15 & -2 & -1
\end{bmatrix} \tag{7.1}
$$

7B. Consider the $[10,32,32]_{\mathbb{Z}}$ formula constructed in Theorem (13), by signing consistently the rows of $D_5 = D_{32,32}$ specified by S_5 in (6.2). For each $k = 1, 2, \ldots, 8$, by deleting the $(8+2k)$ columns $17-k, \ldots, 16, 17, \ldots, 24$ and $33-k, \ldots 32$, we obtain a signed intercalate matrix of type $(10, 24-2k, 32-k)$ with the k colors $25-k, \ldots, 24$ deleted. This gives a formula of type $[10, 24-2k, 32-k]_{\mathbb{Z}}$. With $k = 5, 4, 3$ respectively, this explains entries $(10;14)$, $(10;16)$ in Theorem (1), and entry $(10;18)$ in Theorem (2). Note that this construction only yields a $[10,22,31]_{\mathbb{Z}}$ formula (with $k=1$). We have, however, a $[10,22,30]_{\mathbb{Z}}$ as explained in Remark 11.

7C. There is a $[12,12,26]_{\mathbb{Z}}$ formula which was known to T.Kirkman in the 1840's. We present this by signing the $(12,12,26)$ in (7.2) below. This matrix contains two obvious submatrices of type $(10,10,16)$, which we sign as in (7.1), with obvious relabelling of colors. It is then an easy matter to sign colors $25, 26$ consistently.

$$
\begin{bmatrix}
+1 & +2 & +3 & +4 & +5 & +6 & +7 & +8 & +9 & +10 & +17 & +18 \\
+2 & -1 & +4 & -3 & +6 & -5 & -8 & +7 & +10 & -9 & +18 & -17 \\
+3 & -4 & -1 & +2 & +7 & +8 & -5 & -6 & +11 & +12 & +19 & +20 \\
+4 & +3 & -2 & -1 & +8 & -7 & +6 & -5 & +12 & -11 & +20 & -19 \\
+5 & -6 & -7 & -8 & -1 & +2 & +3 & +4 & +13 & +14 & +21 & +22 \\
+6 & +5 & -8 & +7 & -2 & -1 & -4 & +3 & +14 & -13 & +22 & -21 \\
+7 & +8 & +5 & -6 & -3 & +4 & -1 & -2 & +15 & -16 & +23 & -24 \\
+8 & -7 & +6 & +5 & -4 & -3 & +2 & -1 & +16 & +15 & +24 & +23 \\
+9 & -10 & -11 & -12 & -13 & -14 & -15 & -16 & -1 & +2 & +25 & +26 \\
+10 & +9 & -12 & +11 & -14 & +13 & +16 & -15 & -2 & -1 & +26 & -25 \\
+17 & -18 & -19 & -20 & -21 & -22 & -23 & -24 & -25 & -26 & -1 & +2 \\
+18 & +17 & -20 & +19 & -22 & +21 & +24 & -23 & -26 & +25 & -2 & -1
\end{bmatrix} \tag{7.2}
$$

7D. In (7.3) below, we present a consistently signed $(11,18,32)$. Here, the signing of the $(9,16,16)$ obtained by deleting the bottom 2 rows and columns $9, 10$ follows (6.3). On the other hand, the signing of the $(10,10,16)$ formed by the first 10 columns and by deleting row 9 follows that of $D_{10,10}$ in (7.1), by relabelling colors $9, 10, \ldots, 16$ by colors $17, 18, \ldots, 24$ respectively. Note that the signing of $D_{8,8}$ in both cases agree. Now, it is an easy matter to sign the remaining colors $25, 26, \ldots, 32$ consistently. For $11 \le s \le 16$, the first s columns contain exactly $24 + 2 \circ (s-10)$ colors, precisely the $(11; s)$ entry of the table in Theorem (1).

(7.3)

+1	+2	+3	+4	+5	+6	+7	+8	+17	+18	+9	+10	+11	+12	+13	+14	+15	+16
+2	−1	+4	−3	+6	−5	−8	+7	+18	−17	+10	−9	−12	+11	−14	+13	+16	−15
+3	−4	−1	+2	+7	+8	−5	−6	+19	+20	+11	+12	−9	−10	−15	−16	+13	+14
+4	+3	−2	−1	+8	−7	+6	−5	+20	−19	+12	−11	+10	−9	−16	+15	−14	+13
+5	−6	−7	−8	−1	+2	+3	+4	+21	+22	+13	+14	+15	+16	−9	−10	−11	−12
+6	+5	−8	+7	−2	−1	−4	+3	+22	−21	+14	−13	+16	−15	+10	−9	+12	−11
+7	+8	+5	−6	−3	+4	−1	−2	+23	−24	+15	−16	−13	+14	+11	−12	−9	+10
+8	−7	+6	+5	−4	−3	+2	−1	+24	+23	+16	+15	−14	−13	+12	+11	−10	−9
+9	−10	−11	−12	−13	−14	−15	−16	−25	−26	−1	+2	+3	+4	+5	+6	+7	+8
+17	−18	−19	−20	−21	−22	−23	−24	−1	+2	+25	+26	+27	+28	+29	+30	+31	+32
+18	+17	−20	+19	−22	+21	+24	−23	−2	−1	+26	−25	+28	−27	+30	−29	+32	−31

7E. Consider the $[18,17,32]_{\mathbb{Z}}$ constructed in (5.3). Deleting the bottom row and moving the rightmost column to the middle of the matrix, we obtain the consistently signed $(17,17,32)$ in (7.4) below. For $12 \le r, s \le 16$, the $r \times s$ submatrix in the upper left corner contains exactly $24 + (r-9) \circ (s-9)$ colors. This is precisely the $(r;s)$ entry in Theorem (1), except for $(r;s) = (12;12), (12;14)$ and $(14;12)$.

(7.4)

+1	+2	+3	+4	+5	+6	+7	+8	+17	+9	+10	+11	+12	+13	+14	+15	+16
+2	−1	+4	−3	+6	−5	−8	+7	+18	−10	+9	−12	+11	−14	+13	+16	−15
+3	−4	−1	+2	+7	+8	−5	−6	+19	−11	+12	+9	−10	−15	−16	+13	+14
+4	+3	−2	−1	+8	−7	+6	−5	+20	−12	−11	+10	+9	−16	+15	−14	+13
+5	−6	−7	−8	−1	+2	+3	+4	+21	−13	+14	+15	+16	+9	−10	−11	−12
+6	+5	−8	+7	−2	−1	−4	+3	+22	−14	−13	+16	−15	+10	+9	+12	−11
+7	+8	+5	−6	−3	+4	−1	−2	+23	−15	−16	−13	+14	+11	−12	+9	+10
+8	−7	+6	+5	−4	−3	+2	−1	+24	−16	+15	−14	−13	+12	+11	−10	+9
+9	+10	+11	+12	+13	+14	+15	+16	+25	−1	−2	−3	−4	−5	−6	−7	−8
+17	−18	−19	−20	−21	−22	−23	−24	−1	−25	−26	−27	−28	−29	−30	−31	−32
+18	+17	−20	+19	−22	+21	+24	−23	−2	+26	−25	+28	−27	+30	−29	−32	+31
+19	+20	+17	−18	−23	−24	+21	+22	−3	+27	−28	−25	+26	+31	+32	−29	−30
+20	−19	+18	+17	−24	+23	−22	+21	−4	+28	+27	−26	−25	+32	−31	+30	−29
+21	+22	+23	+24	+17	−18	−19	−20	−5	+29	−30	−31	−32	−25	+26	+27	+28
+22	−21	+24	−23	+18	+17	+20	−19	−6	+30	+29	−32	+31	−26	−25	−28	+27
+23	−24	−21	+22	+19	−20	+17	+18	−7	+31	+32	+29	−30	−27	+28	−25	−26
+24	+23	−22	−21	+20	+19	−18	+17	−8	+32	−31	+30	+29	−28	−27	+26	−25

7F. The construction of $[r, s, n]_{\mathbb{Z}}$ formulae realizing the data in Theorem (1) is now complete by noting that we have indeed constructed a $[12, 12, 26]_{\mathbb{Z}}$ in 7C, and $[12, 14, 30]_{\mathbb{Z}}$, $[14, 12, 30]_{\mathbb{Z}}$ in Remark 12. For an alternative construction of a $[12, 14, 30]_{\mathbb{Z}}$, see [Y5].

8. Construction of $[12, 32, 52]_{\mathbb{Z}}$

Recall the construction of formulae of the Hurwitz-Radon types in Theorem (10). For $t = 6$, we obtain a $[12, 64, 64]$ formula by consistently signing the intercalate matrix M consisting of the following rows of $D_{64,64}$:

1, 2, 3, 4, 5, 6, 9, 10, 17, 18, 33, 34. The signing of this matrix being consistent when restricted to any of its submatrices, we shall disregard the signs and focus on the enumeration of colors in the submatrices of M. This intercalate matrix M admits an obvious *contraction* by 2×2 matrices into $M' \otimes (2, 2, 2)$, (see [Y3]), where M' can be taken to be the matrix consisting of rows 1, 2, 3, 5, 9, 17 of $D_{32,32}$. In this intercalate matrix M', the colors 16, 24, 28, 30, 31, 32 appear in a totality of 16 columns. By deleting these 16 columns one obtains the following $(6, 16, 26)$:

$$\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 9 & 10 & 11 & 13 & 17 & 18 & 19 & 21 & 25\\
2 & 1 & 4 & 3 & 6 & 5 & 8 & 10 & 9 & 12 & 14 & 18 & 17 & 20 & 22 & 26\\
3 & 4 & 1 & 2 & 7 & 8 & 5 & 11 & 12 & 9 & 15 & 19 & 20 & 17 & 23 & 27\\
5 & 6 & 7 & 8 & 1 & 2 & 3 & 13 & 14 & 15 & 9 & 21 & 22 & 23 & 17 & 29\\
9 & 10 & 11 & 12 & 13 & 14 & 15 & 1 & 2 & 3 & 5 & 25 & 26 & 27 & 29 & 17\\
17 & 18 & 19 & 20 & 21 & 22 & 23 & 25 & 26 & 27 & 29 & 1 & 2 & 3 & 5 & 9
\end{bmatrix}$$

This shows that M contains a submatrix of type $(12, 32, 52)$, explaining the entry $(12; 32)$ in Theorem (2). We remark that similar restrictions of M yield formulae of types $[12, 34, 56]_{\mathbb{Z}}$, $[12, 38, 58]_{\mathbb{Z}}$, $[12, 44, 60]_{\mathbb{Z}}$ and $[12, 52, 62]_{\mathbb{Z}}$.

9. Proof of Theorem (2)

Theorem (2) is established by an explicit construction, for each entry with a bullet, a formula of the type indicated by the entry. We shall freely make use of (optimal) formulae of type $[r, s, n]_{\mathbb{Z}}$ for $r, s \leq 16$, given by Theorem (1), explicitly constructed in Theorem (13) and Section 7. These formulae all have integer coefficients. For convenience, we shall write $[r, s, n]$ in place of $[r, s, n]_{\mathbb{Z}}$.

(1) $[10, 18, 29]$ has been constructed in Section 7B.
(2) $[10, 22, 30]$ is explained in Remark 11.
(3) $[10, 32, 32]$ is of the Hurwitz-Radon type.
(4) $[11, 25, 46] = [11, 16, 30] \oplus [11, 9, 16]$.
(5) $[12, 20, 32]$ is explained in Remark 12.
(6) $[12, 21, 42] = [12, 12, 26] \oplus [12, 9, 16]$.
(7) $[12, 20 + k, 32 + (12 *_{\mathbb{Z}} k)] = [12, 20, 32] \oplus [12, k, 12 *_{\mathbb{Z}} k]$. For $k = 4, 9$, this gives $[12, 24, 44]_{\mathbb{Z}}$ and $[12, 29, 48]_{\mathbb{Z}}$ respectively.
(8) $[12, 32, 52]$ has been constructed in Section 8.
(9) $[13, 13 + 9k, 28 + 16k] = [13, 13, 28] \oplus k[13, 9, 16]$ for $k = 1, 2$. This gives $[13, 22, 44]$ and $[13, 31, 60]$.
(10) $[14, s + 9k, (14 *_{\mathbb{Z}} s) + 16k] = [14, s, 14 *_{\mathbb{Z}} s] \oplus k[14, 9, 16]$ for $s = 10, 11, 12$ and $k = 1, 2$. This gives $[14, 19, 43]$, $[14, 20, 44]$, $[14, 21, 46]$, $[14, 28, 59]$, $[14, 29, 60]$, and $[14, 30, 62]$.
(11) $[16, 19, 44] = [16, 10, 28] \oplus [16, 9, 16]$.
(12) $[16, 20, 46] = [16, 11, 30] \oplus [16, 9, 16]$.
(13) $[16, 18 + k, 32 + (16 *_{\mathbb{Z}} k)] = [16, 18, 32] \oplus [16, k, 16 *_{\mathbb{Z}} k]$ for $k = 9, 10, 11$. This gives $[16, 27, 48]$, $[16, 28, 60]$ and $[16, 29, 62]$.

(14) $[17, 18, 32]$ has been constructed by Adem [A1], see (5.3).
(15) $[17, s, 30 + s] = [17, 18, 32] \oplus ([16, s - 18, 16] \oplus' [1, s - 18, s - 18])$ for $19 \leq s \leq 27$.
(16) $[17, 28, 61] = [17, 18, 32] \oplus [17, 10, 29]$.
(17) $[18, 17 + k, 48 + (2 \circ k)] = [18, 17, 32] \oplus ([16, k, 16] \oplus' [2, k, 2 \circ k])$ for $1 \leq k \leq 9$. In particular, this gives $[18, 19, 50]$, $[18, 21, 52]$, $[18, 23, 54]$, $[18, 25, 56]$ and $[18, 26, 58]$.
(18) $[18, 27, 61] = [18, 17, 32] \oplus [18, 10, 29]$.
(19) $[19, 21, 59] = [10, 21, 30] \oplus' [9, 21, 29]$.
(20) $[20, 20, 56] = [12, 20, 32] \oplus' [8, 20, 24]$.
(22) $[20, 23, 60] = \mathfrak{D}'[10, 22, 30]$.
(22) $[20, 32, 64] = 2[10, 32, 32]$. In fact, there is $[20, 33, 64] = \mathfrak{D}'[10, 32, 32]$.
(23) $[21, 24, 64] = \mathfrak{D}[20, 12, 32]$.
(24) $[r, 25, r+54] = [r, 8, 24] \oplus ([18, 17, 32] \oplus' ([r-18, 16, 16] \oplus [r-18, 1, r-18]))$ for $21 \leq r \leq 24$.
(25) $[17+h, 26, 72+(2 \circ h)] = [17+h, 8, 24] \oplus ([17, 18, 32] \oplus [h, 18, 16+(2 \circ h)])$ for $1 \leq h \leq 7$. For $h = 4, 6$, this gives $[21, 26, 76]$ and $[23, 26, 78]$ respectively.
(26) $[22, 29, 80] = ([12, 20, 32] \oplus [12, 9, 16]) \oplus' [10, 29, 32]$.
(27) $[22, 32, 84] = [12, 32, 52] \oplus [10, 32, 32]$.
(28) $[23, 28, 84] = [23, 8, 24] \oplus \mathfrak{D}[22, 10, 30]$.
(29) $[23, 31, 92] = [13, 31, 60] \oplus' [10, 31, 32]$.
(30) $[24, 24, 72] = 3[8, 24, 24]$.
(31) $[24, 29, 88] = [24, 8, 24] \oplus \mathfrak{D}'[12, 20, 32]$.
(32) $[24, 30, 94] = [14, 30, 62] \oplus' [10, 30, 32]$.
(33) $[17+h, 28, 80+(2 \circ h)] = [17+h, 10, 32] \oplus ([17, 18, 32] \oplus' [h, 18, 16+(2 \circ h)])$ for $h = 8, 9$. This gives $[25, 28, 88]$ and $[26, 28, 90]$.
(34) $[26, 27, 80] = 3[16, 9, 16] \oplus' [10, 27, 32]$.
(35) $[26, 29, 94] = [16, 29, 62] \oplus' [10, 29, 32]$.
(36) $[27, 27, 89] = [17, 27, 57] \oplus' [10, 27, 32]$
(37) $[27, 28, 93] = [17, 28, 61] \oplus' [10, 28, 32]$.
(38) $[30, 32, 96] = 3[10, 32, 32]$.
(39) $[32, 32, 116] = [12, 32, 52] \oplus' [20, 32, 64]$.

This completes the justification of the data of Theorem (2). The fact that those entries with asterisks give *precise* values of $r *_{\mathbb{Z}} s$ follows from Proposition (8).

10. Comparison with previous upper bounds

Adem [A1] has constructed a number of $[r, s, n]_{\mathbb{Z}}$ formulae arising from the restriction of the multiplication of the Cayley-Dickson algebras. In the range considered in Theorem (2), the following entries gives better upper bounds the $r *_{\mathbb{Z}} s$ than Adem: (13; 24) by 4, (14; 26) by 8, (19; 20) by 2, (22; 23) by 14, (23; 23) by 22, (23; 24) by 24, (25; 26) by 24, (27, 28) by 19. For the entries (21; 22), (18; 19), (19; 19), Adem obtained the same upper bounds as in Theorem (2). Formulae of types $[21, 24, 64]$, $[24, 29, 88]$, $[24, 24, 72]$ and

[30, 32, 96] can also be found in [LS]. We conclude with a comparison of the upper bounds of $r *_{\mathbb{Z}} r$, $18 \le r \le 32$, obtained in [A1],[LS], and Theorem (2).

	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
[A1]	50	56	64	64	86	94	104	104	112	112	128	128	128	128	128
[LS]	50	56	60	64	72	72	72	92	92	96	96	96	96	120	128
New	50	56	56	64	72	72	72	80	80	89	96	96	96	116	116

The authors would like to thank the referee for his many suggestions on improvement of the presentation of this paper.

DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF CINCINNATI
CINCINNATI, OHIO 45221-0025

DEPARTMENT OF MATHEMATICS
FLORIDA ATLANTIC UNIVERSITY
BOCA RATON, FLORIDA 33431-0991
U.S.A.

REFERENCES

[A1] J. ADEM, *Construction of some normed maps,* Bol. Soc. Mat. Mexicana, **20,** (1975) 59–75.

[A2] ———— *On the Hurwitz problem over an arbitrary field I,* Bol. Soc. Mat. Mexicana, **25,** (1980) 29–51.

[A3] ———— *On the Hurwitz problem over an arbitrary field II,* Bol. Soc. Mat. Mexicana, **26,** (1981) 29–41.

[E] B. ECKMANN, *Gruppentheoretischer Beweis des Satzes von Hurwitz-Radon über die Komposition der quadratischen Formen,* Comment. Math. Helv. **15,** 1942–43 358–366.

[F] N. J. FINE, *Binomial coefficients modulo a prime,* Amer. Math. Monthly, **54,** (1947) 589–592.

[GP] A. V. GERAMITA AND N. PULLMAN, *A theorem of Hurwitz and Radon and orthogonal projective modules,* Proc. Amer. Math. Soc. **42,** (1974) 51–56.

[H] A. HURWITZ, *Über die Komposition der quadratischen Formen,* Math. Ann. **88,** (1923) 1–25.

[L1] K. Y. LAM, *Thesis,* Princeton University, 1966.

[L2] ———— *Construction of nonsingular bilinear maps,* Topology **6,** (1967) 423–426.

[L3] ———— *Topological methods for studying the composition of quadratic forms,* Canad. Math. Soc. Conf. Proc. **4,** (1984) 173–192.

[L4] ———— *Some new results on composition of quadratic forms,* Invent. Math. **79,** (1985) 467–474.

LY1] ———— AND P. YIU, *Sums of squares formulae near the Hurwitz–Radon range,* Contemp. Math. **58** (part II) (1987) 51–56.

[LY2] ———— AND ———— *Geometry of normed bilinear maps and the 16–square problem,* Math. Ann. **284** (1989) 437–447.

[LY3] ———— AND *Retrieving hidden sums of squares formulae ,* preprint.

[LS] T. Y. LAM AND T. L. SMITH, *On Yuzvinsky's monomial pairings,* to appear in Quart. J. Math. Oxford.

[P] A. PFISTER, *Zur Darstellung von -1 als Summe von Quadraten in einem Körper,* J. London Math. Soc. **40,** (1965) 159–165.

[R] J. RADON, *Lineare Scharen orthogonaler Matrizen,* Abh. Math. Sem. Univ. Hamburg **1,** (1922) 1–14.

[S1] D. B. SHAPIRO, *Spaces of similarities I,* J. Algebra **46,** (1977) 148–170.

[S2] ________ *Products of sums of squares,* Expo. Math. **2,** (1984) 235–261.

[S3] ________ *Compositions of quadratic forms,* forthcoming.

[SE] N. E. STEENROD AND D. B. A. EPSTEIN, *Cohomology Operations,* Annals of Math. Studies, **50,** Princeton University Press (1962).

[W] Y. C. WONG, *Isoclinic n-planes in Euclidean 2n-space, Clifford parallels in Elliptic* $(2n-1)$-*space, and the Hurwitz matrix equations,* Mem. Amer. Math. Soc. **41,** (1961)

[Y1] P. YIU, *Thesis,* University of British Columbia, (1985).

[Y2] ________ *Quadratic forms between spheres and the nonexistence of sums of squares formulae,* Math. Proc. Camb. Phil. Soc. **100,** (1986) 493–504.

[Y3] ________ *Sums of squares formulae with integer coefficients,* Canad. Math. Bull. **30,** (1987) 318–324.

[Y4] ________ *On the product of two sums of 16 squares as a sum of squares of integral bilinear forms,* Quart. J. Math. Oxford (2) **41,** (1990) 463–500.

[Y5] ________ *Composition of sums of squares with integer coefficients,* to appear in "Deformations of Mathematical Structures. Hurwitz-type structures and applications to surface physics" (ed. J.Lawrynowicz et al.)

[Yuz1] S. YUZVINSKY, *Orthogonal pairings of euclidean spaces,* Michigan Math.J. **28,** (1981) 131–145.

[Yuz2] ________ *A series of monomial pairings,* Linear and Multlinear Algebra, **15,** (1984) 109–119.

[Z] P. ZVENGROWSKI, *Canonical vector fields on spheres,* Comment. Math. Helv. **43,** (1968) 341–347.